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ПОЛИНОМИАЛЬНЫЕ АВТОМОРФИЗМЫ, КВАНТОВАНИЕ И ЗАДАЧИ ВОКРУГ ГИПОТЕЗЫ ЯКОБИАНА. III. АВТОМОРФИЗМЫ, ТОПОЛОГИЯ ПОПОЛНЕНИЯ И АППРОКСИМАЦИЯ

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Аннотация. Работа является третьей частью обзора результатов, касающихся квантового подхода к некоторым классическим аспектам некоммутативных алгебр. Первая часть: Итоги науки и техники. Современная математика и ее приложения. Тематические обзоры. — 2022. — *213.* — С. 110–144. Вторая часть: Итоги науки и техники. Современная математика и ее приложения. Тематические обзоры. — 2022. — *214.* — С. 107–126. Продолжение будет опубликовано в следующих выпусках.

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POLYNOMIAL AUTOMORPHISMS, QUANTIZATION, AND JACOBIAN CONJECTURE RELATED PROBLEMS. III. AUTOMORPHISMS, AUGMENTATION TOPOLOGY, AND APPROXIMATION

© 2022 **A. M. ELISHEV, A. Ya. KANEL-BELOV, F. RAZAVINIA, J.-T. YU, W. ZHANG**

Abstract. This paper is the third part of a review of results concerning the quantization approach to the some classical aspects of noncommutative algebras. The first part is: Itogi Nauki Tekhn. Sovr. Mat. Prilozh. Temat. Obzory, **213** (2022), pp. 110–144. The second part is: Itogi Nauki Tekhn. Sovr. Mat. Prilozh. Temat. Obzory, **214** (2022), pp. 107–126. Continuation will be published in future issues.

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CONTENTS

Chapter 3

AUTOMORPHISMS, AUGMENTATION TOPOLOGY, AND APPROXIMATION

3.1. Introduction and Main Results

This chapter is dedicated to the review of results of Kanel-Belov, Yu and Elishev on the geometry of the Ind-schemes $\operatorname{Aut}\mathbb{K}[x_1,\dots,x_n]$ and $\operatorname{Aut}\mathbb{K}\langle x_1,\dots,x_n\rangle$ of automorphisms of the polynomial algebra and the free associative algebra over an algebraically closed field, with the number of generators >2. The inner character of Ind automorphisms of these Ind-schemes, together with the negative resolution of the automorphism group lifting problem, was established in [123].

3.1.1. Automorphisms of $K[x_1,\dots,x_n]$ and $K\langle x_1,\dots,x_n\rangle$. Let K be a field. The main objects of this study are the K-algebra automorphism groups $\operatorname{Aut}K[x_1,\dots,x_n]$ and $\operatorname{Aut}K\langle x_1,\dots,x_n\rangle$ of the (commutative) polynomial algebra and the free associative algebra with n generators, respectively. The former is equivalent to the group of all polynomial one-to-one mappings of the affine space $\mathbb{A}^n_K$. Both groups admit a representation as a colimit of algebraic sets of automorphisms filtered by total degree (with morphisms in the direct system given by closed embeddings) which turns them into topological spaces with Zariski topology compatible with the group structure. The two groups carry a power series topology as well, since every automorphism φ may be identified with the n-tuple $(\varphi(x_1),\dots,\varphi(x_n))$ of the images of generators. This topology plays an especially important role in the applications, and it turns out—as reflected in the main results of this study—that approximation properties arising from this topology agree well with properties of combinatorial nature.

Ind-Groups of polynomial automorphisms play a central part in the study of the Jacobian conjecture of O. Keller as well as a number of problems of similar nature. One outstanding example is provided by a recent conjecture of Kanel-Belov and Kontsevich (BKK conjecture; see [41, 42], which asks whether the group

$$\mathrm{Sympl}(\mathbb{C}^{2n})\subset \mathrm{Aut}(\mathbb{C}[x_1,\dots,x_{2n}])$$

of complex polynomial automorphisms preserving the standard Poisson bracket

$$\{x_i,\ x_j\}=\delta_{i,n+j}-\delta_{i+n,j}$$

is isomorphic[1] to the group of automorphisms of the nth Weyl algebra W_n

$$W_n(\mathbb{C})=\mathbb{C}\big\langle x_1,\dots,x_n,y_1,\dots,y_n\big\rangle/I,$$
$$I=\big(x_ix_j-x_jx_i,\ y_iy_j-y_jy_i,\ y_ix_j-x_jy_i-\delta_{ij}\big).$$

[1] In fact, the conjecture seeks to establish an isomorphism $\mathrm{Sympl}(K^{2n})\simeq \mathrm{Aut}(W_n(K))$ for any field K of characteristic zero in a functorial manner.

The physical meaning of the Kanel-Belov–Kontsevich conjecture is the invariance of the polynomial symplectomorphism group of the phase space under the procedure of deformation quantization.

The BKK conjecture was conceived during a successful search for a proof of stable equivalence of the Jacobian conjecture and a well-known conjecture of Dixmier stating that $\mathrm{Aut}(W_n) = \mathrm{End}(W_n)$ over any field of characteristic zero. In the papers [41,42], a particular family of homomorphisms (in effect, monomorphisms) $\mathrm{Aut}(W_n(\mathbb{C})) \to \mathrm{Sympl}(\mathbb{C}^{2n})$ was constructed, and a natural question whether those homomorphisms were in fact isomorphisms was raised. The aforementioned morphisms, independently studied by Tsuchimoto to the same end, were in actuality defined as restrictions of morphisms of the saturated model of Weyl algebra over an algebraically closed field of *positive* characteristic—an object which contains $W_n(\mathbb{C})$ as a proper subalgebra. One of the defined morphisms turned out to have a particularly simple form over the subgroup of the so-called tame automorphisms, and it was natural to assume that morphism was the desired BKK isomorphism (at least for the case of algebraically closed base field). Central notion of the construction is the notion of infinitely large prime number (in the sense of hyperintegers), which arises as the sequence $(p_m)_{m\in\mathbb{N}}$ of positive characteristics of finite fields comprising the saturated model. This leads to the following natural problem (see [41]).

Problem 3.1.1. Prove that the BKK morphism is independent of the choice of the infinite prime $(p_m)_{m\in\mathbb{N}}$.

The general formulation of this question presented in [41] is as follows.

For a commutative ring R define

$$R_\infty = \varinjlim \left(\prod_p R' \otimes \mathbb{Z}/p\mathbb{Z} \Big/ \bigoplus_p R' \otimes \mathbb{Z}/p\mathbb{Z} \right),$$

where the direct limit is taken over the filtered system of all finitely generated subrings $R' \subset R$ and the product and the sum are taken over all primes p. This larger ring possesses a unique "nonstandard Frobenius" endomorphism

$$\mathrm{Fr} : R_\infty \to R_\infty, \quad (a_p)_{\text{primes } p} \mapsto (a_p^p)_{\text{primes } p}.$$

The Kanel-Belov–Kontsevich construction returns a morphism

$$\psi_R : \mathrm{Aut}(W_n(R)) \to \mathrm{Sympl}\, R_\infty^{2n}$$

such that there exists a unique homomorphism

$$\phi_R : \mathrm{Aut}(W_n)(R) \to \mathrm{Aut}(P_n)(R_\infty)$$

satisfying the condition $\psi_R = \mathrm{Fr}_* \circ \phi_R$, where $\mathrm{Fr}_* : \mathrm{Aut}(P_n)(R_\infty) \to \mathrm{Aut}(P_n)(R_\infty)$ is the Ind-group homomorphism induced by the Frobenius endomorphism of the coefficient ring and P_n is the commutative Poisson algebra, i.e., the polynomial algebra in $2n$ variables equipped with additional Poisson structure (so that $\mathrm{Aut}(P_n(R))$ is just the group $\mathrm{Sympl}(R^{2n})$ of Poisson structure-preserving automorphisms).

Question 3.1.2. In the above formulation, does the image of ϕ_R belong to

$$\mathrm{Aut}(P_n)(i(R) \otimes \mathbb{Q}),$$

where $i : R \to R_\infty$ is the tautological inclusion? In other words, does there exist a unique homomorphism

$$\phi_R^{\mathrm{can}} : \mathrm{Aut}(P_n)(R) \to \mathrm{Aut}(P_n)(R \otimes \mathbb{Q})$$

such that $\psi_R = \mathrm{Fr}_* \circ i_* \circ \phi_R^{\mathrm{can}}$.

Comparing the two morphisms ϕ and φ defined using two different free ultrafilters, we obtain a "loop" element $\phi\varphi^{-1}$ of $\mathrm{Aut}_{\mathrm{Ind}}(\mathrm{Aut}(W_n))$ (i.e., an automorphism which preserves the structure of infinite dimensional algebraic group). Describing this group would provide a solution to this question.

In the spirit of the above we propose the following conjecture.

Conjecture 3.1.3. All automorphisms of the Ind-group $\mathrm{Sympl}(\mathbb{C}^{2n})$ are inner.

A similar conjecture may be stated for $\mathrm{Aut}(W_n(\mathbb{C}))$.

The automorphism groups of Weyl algebras and their generalizations, as well as automorphisms of certain algebras of vector fields, were studied in the works of Bavula [23–25]. Reduction to positive characteristic was proved both fruitful and essential in the context of Weyl algebra. One of the precursors to the study of these algebras in characteristic p was the paper [22].

We are focused on the investigation of the group $\mathrm{Aut}(\mathrm{Aut}(K[x_1, \dots, x_n]))$ and the corresponding noncommutative (free associative algebra) case. This way of thinking has its roots in the realm of universal algebra and universal algebraic geometry and was conceived in the pioneering work of Boris Plotkin. A more detailed discussion can be found in [39].

Wild automorphisms and the lifting problem. In 2004, the celebrated Nagata conjecture over a field K of characteristic zero was proved by Shestakov and Umirbaev (see [180, 183]) and a stronger version of the conjecture was proved by Umirbaev and Yu (see [200]). Let K be a field of characteristic zero. Every wild $K[z]$-automorphism (wild $K[z]$-coordinate) of $K[z][x, y]$ is wild viewed as a K-automorphism (K-coordinate) of $K[x, y, z]$. In particular, the Nagata automorphism $\big(x - 2y(y^2 + xz) - (y^2 + xz)^2 z,\ y + (y^2 + xz)z, z\big)$ (the Nagata coordinates $x - 2y(y^2 + xz) - (y^2 + xz)^2 z$ and $y + (y^2 + xz)z$) are wild. In [200], the following related question was raised.

The lifting problem. *Can an arbitrary wild automorphism (wild coordinate) of the polynomial algebra $K[x, y, z]$ over a field K be lifted to an automorphism (coordinate) of the free associative algebra $K\langle x, y, z\rangle$?*

In the paper [38] based on the degree estimate (see [137, 143]), Belov-Kanel and Yu proved that any wild z-automorphism including the Nagata automorphism cannot be lifted as a z-automorphism (moreover, it was proved in [44] that every z-automorphism of $K\langle x, y, z\rangle$ is stably tame and becomes tame after adding at most one variable). It means that if every automorphism can be lifted, then it provides an obstruction z' to z-lifting and the question to estimate such an obstruction is naturally raised.

In view of the above, we may state the following problem.

The automorphism group lifting problem. *Is $\mathrm{Aut}(K[x_1, \dots, x_n])$ isomorphic to a subgroup of $\mathrm{Aut}(K\langle x_1, \dots, x_n\rangle)$ under the natural abelianization?*

The following examples show this problem is interesting and nontrivial.

Example 3.1.4. There is a surjective homomorphism (taking the absolute value) from $\mathbb{C}^*$ onto $\mathbb{R}^+$. But $\mathbb{R}^+$ is isomorphic to the subgroup $\mathbb{R}^+$ of $\mathbb{C}^*$ under the homomorphism.

Example 3.1.5. There is a surjective homomorphism (taking the determinant) from $\mathrm{GL}_n(\mathbb{R})$ onto $\mathbb{R}^*$. But obviously $\mathbb{R}^*$ is isomorphic to the subgroup $\mathbb{R}^* I_n$ of $\mathrm{GL}_n(\mathbb{R})$.

In this paper we prove that the automorphism group lifting problem has a negative answer.

The lifting problem and the automorphism group lifting problem are closely related to the Kanel-Belov–Kontsevich conjecture (see Sec. 3.3.1).

Consider a symplectomorphism $\varphi : x_i \to P_i$, $y_i \to Q_i$. It can be lifted to some automorphism $\widehat{\varphi}$ of the quantized algebra $W_\hbar[[\hbar]]$:

$$\widehat{\varphi} : x_i \to P_i + P_i^1 \hbar + \dots + P_i^m \hbar^m;\ y_i \to Q_i + Q_i^1 \hbar + \dots + Q_i^m \hbar^m.$$

The point is to choose a lift $\widehat{\varphi}$ in such a way that the degree of all P_i^m and Q_i^m would be bounded. If this is true, then the BKK conjecture is also true.

3.1.2. Main results.

The main results of this paper are as follows.

Theorem 3.1.6. *Any* Ind*-scheme automorphism φ of* $\mathrm{NAut}(K[x_1, \dots, x_n])$ *for $n \geqslant 3$ is inner, i.e., is a conjugation via some automorphism.*

Theorem 3.1.7. *Any* Ind*-scheme automorphism φ of* $\mathrm{NAut}(K\langle x_1, \dots, x_n\rangle)$ *for $n \geqslant 3$ is semi-inner* (see Definition 3.1.11).

We denote by NAut the group of *nice* automorphisms, i.e., automorphisms that can be approximated by tame automorphisms (see Definition 3.3.2). In the case of zero characteristic, every automorphism is nice.

For the group of automorphisms of a semigroup a number of similar results on set-theoretical level was obtained previously by Kanel-Belov, Lipyanski, and Berzinsh (see [39, 43]). All these questions (including Aut(Aut) investigation) take root in the realm of Universal Algebraic Geometry and were proposed by B. Plotkin. Equivalence of two algebras having the same generalized identities and isomorphism of first order means semi-inner properties of automorphisms (for details, see [39, 43]).

Automorphisms of tame automorphism groups. Regarding the tame automorphism group, something can be done on the group- theoretic level. In the paper of H. Kraft and I. Stampfli (see [133]) the automorphism group of the tame automorphism group of the polynomial algebra was thoroughly studied. In [133], conjugation of elementary automorphisms via translations plays an important role. The results of our study are different. We describe the group $\mathrm{Aut}(\mathrm{TAut}_0)$ of the group TAut_0 of tame automorphisms preserving the origin (i.e., taking the augmentation ideal onto an ideal which is a subset of the augmentation ideal). This is technically more difficult, and will be universally and systematically done for both commutative (polynomial algebra) case and noncommutative (free associative algebra) case. We observe a few problems in the shift conjugation approach for the non-commutative (free associative algebra) case, as it was for commutative case in [133]. Any evaluation on a ground field element can return zero, for example, in the Lie polynomial $[[x, y], z]$. Note that the calculations of $\mathrm{Aut}(\mathrm{TAut}_0)$ (respectively, $\mathrm{Aut}_{\mathrm{Ind}}(\mathrm{TAut}_0)$ and $\mathrm{Aut}_{\mathrm{Ind}}(\mathrm{Aut}_0)$) imply also the same results for $\mathrm{Aut}(\mathrm{TAut})$ (respectively, $\mathrm{Aut}_{\mathrm{Ind}}(\mathrm{TAut})$ and $\mathrm{Aut}_{\mathrm{Ind}}(\mathrm{Aut})$) according to the approach of this article via stabilization by the torus action.

Theorem 3.1.8. *Any automorphism φ of $\mathrm{TAut}_0(K[x_1, \ldots, x_n])$ (in the group-theoretic sense) for $n \geqslant 3$ is inner, i.e. is a conjugation via some automorphism.*

Theorem 3.1.9. *The group $\mathrm{TAut}_0(K[x_1, \ldots, x_n])$ is generated by the automorphism*

$$x_1 \to x_1 + x_2x_3, \quad x_i \to x_i, \quad i \neq 1,$$

and linear substitutions if $\mathrm{char}(K) \neq 2$ and $n > 3$.

Let $G_N \subset \mathrm{TAut}(K[x_1, \ldots, x_n])$ and $E_N \subset \mathrm{TAut}(K\langle x_1, \ldots, x_n\rangle)$ be tame automorphism subgroups preserving the Nth power of the augmentation ideal.

Theorem 3.1.10. *Any automorphism φ of G_N (in the group-theoretic sense) for $N \geqslant 3$ is inner, i.e., is given by a conjugation via some automorphism.*

Definition 3.1.11. An *anti-automorphism* Ψ of a K-algebra B is a vector-space automorphism such that $\Psi(ab) = \Psi(b)\Psi(a)$. For example, transposition of matrices is an anti-automorphism. An anti-automorphism of the free associative algebra A is a *mirror anti-automorphism* if it sends x_ix_j to x_jx_i for some fixed i and j. If a mirror anti-automorphism θ acts identical on all generators x_i, then for any monomial $x_{i_1} \cdots x_{i_k}$ we have

$$\theta(x_{i_1} \cdots x_{i_k}) = x_{i_k} \cdots x_{i_1}.$$

Such an anti-automorphism will be generally referred to as *the* mirror anti-automorphism.

An automorphism of $\mathrm{Aut}(A)$ is *semi-inner* if it can be expressed as a composition of an inner automorphism and a conjugation by a mirror anti-automorphism.

Theorem 3.1.12.

(a) *Any automorphisms φ of $\mathrm{TAut}_0(K\langle x_1, \ldots, x_n\rangle)$ and also $\mathrm{TAut}(K\langle x_1, \ldots, x_n\rangle)$ (in the group-theoretic sense) for $n \geqslant 4$ are semi-inner, i.e., are conjugations via some automorphism and/or mirror anti-automorphism.*
(b) *The same is true for E_n, $n \geqslant 4$.*

The case of $\mathrm{TAut}(K\langle x, y, z\rangle)$ is substantially more difficult. We can treat it only on Ind-scheme level, but even then it is the most technical part of the paper (see Sec. 3.5.2). For the two-variable case, a similar proposition is probably invalid.

Theorem 3.1.13.

(a) *Let* $\mathrm{char}(K) \neq 2$. *Then* $\mathrm{Aut}_{\mathrm{Ind}}(\mathrm{TAut}(K\langle x, y, z\rangle))$ (*respectively,* $\mathrm{Aut}_{\mathrm{Ind}}(\mathrm{TAut}_0(K\langle x, y, z\rangle))$) *is generated by conjugation by an automorphism or a mirror anti-automorphism.*

(b) *The same is true for* $\mathrm{Aut}_{\mathrm{Ind}}(E_3)$.

We denote by TAut the tame automorphism group and by $\mathrm{Aut}_{\mathrm{Ind}}$ the group of Ind-scheme automorphisms (see Sec. 3.2.2).

Approximation allows us to formulate the celebrated Jacobian conjecture for any characteristic.

Lifting of the automorphism groups. In this paper, we prove that the automorphism group of polynomial algebra over an arbitrary field K cannot be embedded into the automorphism group of free associative algebra induced by the natural abelianization.

Theorem 3.1.14. *Let K be an arbitrary field, $G = \mathrm{Aut}_0(K[x_1, \dots, x_n])$ and $n > 2$. Then G cannot be isomorphic to any subgroup H of $\mathrm{Aut}(K\langle x_1, \dots, x_n\rangle)$ induced by the natural abelianization. The same is true for $\mathrm{NAut}(K[x_1, \dots, x_n])$.*

3.2. Varieties of Automorphisms

3.2.1. Elementary and tame automorphisms.

Let P be a polynomial that is independent of x_i with i fixed. The automorphism

$$x_i \to x_i + P, \quad x_j \to x_j \quad \text{for } i \neq j$$

is called *elementary.* The group generated by linear automorphisms and elementary ones for all possible P is called the *tame automorphism group* (or *subgroup*) TAut and elements of TAut are *tame automorphisms.*

3.2.2. Ind-Schemes and Ind-groups.

Definition 3.2.1. An Ind-*variety* M is the direct limit of algebraic varieties $M = \varinjlim\{M_1 \subseteq M_2 \cdots\}$. An Ind-*scheme* is an Ind-variety which is a group such that the group inversion is a morphism $M_i \to M_{j(i)}$ of algebraic varieties, and the group multiplication induces a morphism from $M_i \times M_j$ to $M_{k(i,j)}$. A map φ is a *morphism* of an Ind-variety M to an Ind-variety N, if $\varphi(M_i) \subseteq N_{j(i)}$ and the restriction φ to M_i is a morphism for all i. Monomorphisms, epimorphisms and isomorphisms are defined similarly in a natural way.

Example 3.2.2. M is the group of automorphisms of the affine space and M_j are the sets of all automorphisms in M with degree $\leqslant j$.

Problem 3.2.3. Investigate growth functions of Ind-varieties. For example, the dimension of varieties of polynomial automorphisms of degree $\leqslant n$.

Note that coincidence of growth functions of $\mathrm{Aut}(W_n(\mathbb{C}))$ and $\mathrm{Sympl}(\mathbb{C}^{2n})$ would imply the Kanel-Belov–Kontsevich conjecture (see [41]).

Definition 3.2.4. The ideal I generated by variables x_i is called the *augmentation ideal.* For a fixed positive integer $N > 1$, the *augmentation subgroup* H_N is the group of all automorphisms φ such that $\varphi(x_i) \equiv x_i \mod I^N$. The larger group $\hat{H}_N \supset H_N$ is the group of automorphisms whose linear part is scalar, and $\varphi(x_i) \equiv \lambda x_i \mod I^N$ (λ is independent of i). We often say an arbitrary element of the group $\hat{H}_N$ is an automorphism that is homothety modulo (the Nth power of) the augmentation ideal.

3.3. Jacobian Conjecture in Any Characteristic, Kanel-Belov–Kontsevich Conjecture, and Approximation

3.3.1. Approximation problems and Kanel-Belov–Kontsevich conjecture. Let us give formulation of the Kanel-Belov–Kontsevich conjecture:

$$\mathbf{BKKC}_n : \mathrm{Aut}(W_n) \simeq \mathrm{Sympl}(\mathbb{C}^{2n}).$$

A similar conjecture can be stated for endomorphisms:

$$\mathbf{BKKC}_n : \mathrm{End}(W_n) \simeq \mathrm{Sympl}\,\mathrm{End}(\mathbb{C}^{2n}).$$

If the Jacobian conjecture JC_{2n} is true, then the respective conjunctions over all n of the two conjectures are equivalent.

It is natural to approximate automorphisms by tame ones. There exists such an approximation up to terms of any order for polynomial automorphisms as well as Weyl algebra automorphisms, symplectomorphisms etc. However, the naive approach fails.

It is known that $\mathrm{Aut}(W_1) \equiv \mathrm{Aut}_1(K[x,y])$ where Aut_1 stands for the subgroup of automorphisms of Jacobian determinant one. However, considerations from [177] show that Lie algebra of the first group is the algebra of derivations of W_1 and thus possesses no identities apart from the ones of the free Lie algebra, another coincidence of the vector fields which diverge to zero, and has polynomial identities. These cannot be isomorphic (see [41, 42]). In other words, this group has two coordinate systems, which are relatively nonsmooth but relatively integral. One system is constructed from the coefficients of differential operators in a fixed basis of generators, while its counterpart is provided by the coefficients of polynomials, which are the images of the basis $\tilde{x}_i, \tilde{y}_i$.

In [177], functionals on $\mathfrak{m}/\mathfrak{m}^2$ were considered in order to define the Lie-algebra structure. In the spirit of this paper, we state the following conjecture.

Conjecture 3.3.1. The natural limit of $\mathfrak{m}/\mathfrak{m}^2$ is zero.

This means that the definition of the Lie algebra admits some sort of functoriality problem and it depends on the presentation of (reducible) Ind-scheme.

In his remarkable paper, Yu. Bodnarchuk established Theorem 3.1.6 by using Shafarevich's results for the tame automorphism subgroup and for the case where the Ind-scheme automorphism is regular in the sense that it sends coordinate functions to coordinate functions (see [55]). In this case, the tame approximation works (as well as for the symplectic case), and the corresponding method is similar to ours. We present it here in order to make the text more self-contained, as well as for the purpose of tackling the noncommutative (that is, the free associative algebra) case. Note that in general, for regular functions, if the Shafarevich-style approximation were valid, then the Kanel-Belov–Kontsevich conjecture would follow directly, which is impossible.

In the sequel, we do not assume regularity in the sense of [55] but only assume that the restriction of a morphism on any subvariety is a morphism again. Note that morphisms of Ind-schemes $\mathrm{Aut}(W_n) \to \mathrm{Sympl}(\mathbb{C}^{2n})$ have this property, but are not regular in the sense of Bodnarchuk (see [55]).

We use the idea of singularity which allows us to prove the augmentation subgroup structure preservation, so that the approximation works in this case.

Consider the isomorphism $\mathrm{Aut}(W_1) \cong \mathrm{Aut}_1(K[x,y])$. It has a strange property. Let us add a small parameter t. Then an element arbitrary close to zero with respect to t^k does not go to zero arbitrarily, so it is impossible to make tame limit! There is a sequence of convergent product of elementary automorphisms, which is not convergent under this isomorphism. Exactly the same situation happens for W_n. These effects cause problems in perturbative quantum field theory.

3.3.2. Jacobian conjecture in an arbitrary characteristic. Recall that the Jacobian conjecture in zero characteristic states that any polynomial endomorphism $\varphi : K^n \to K^n$ with constant Jacobian is globally invertible.

A naive attempt to directly transfer this formulation to positive characteristic fails because of the counterexample $x \mapsto x - x^p$ ($p = \mathrm{char}\, K$), whose Jacobian is everywhere 1 but which is evidently not invertible. Approximation provides a way to formulate a suitable generalization of the Jacobian conjecture to any characteristic and put it in a framework of other questions.

Definition 3.3.2. An endomorphism $\varphi \in \mathrm{End}(K[x_1, \ldots, x_n])$ is *good* if for any m, there exist $\psi_m \in \mathrm{End}(K[x_1, \ldots, x_n])$ and $\phi_m \in \mathrm{Aut}(K[x_1, \ldots, x_n])$ such that

(i) $\varphi = \psi_m \phi_m$;

(ii) $\psi_m(x_i) \equiv x_i \mod (x_1, \ldots, x_n)^m$.

An automorphism $\varphi \in \mathrm{Aut}(K[x_1, \ldots, x_n])$ is *nice* if for any m there exist $\psi_m \in \mathrm{Aut}(K[x_1, \ldots, x_n])$ and $\phi_m \in \mathrm{TAut}(K[x_1, \ldots, x_n])$ such that

(i) $\varphi = \psi_m \phi_m$;

(ii) $\psi_m(x_i) \equiv x_i \mod (x_1, \ldots, x_n)^m$, i.e. $\psi_m \in H_m$.

Anick proved (see [8]) that if $\mathrm{char}(K) = 0$, any automorphism is nice. However, this is unclear in positive characteristic.

Question 3.3.3. Is any automorphism over arbitrary field nice?

Ever good automorphism has Jacobian 1, and all such automorphisms are good—and even nice—if $\mathrm{char}(K) = 0$. This observation allows for the following question to be considered a generalization of the Jacobian conjecture to positive characteristic.

Jacobian conjecture in any characteristic: *Is any good endomorphism over arbitrary field an automorphism?*

Similar notions can be formulated for the free associative algebra. That justifies the following question.

Question 3.3.4. Is any automorphism of free associative algebra over arbitrary field nice?

Question 3.3.5 (version of free associative positive characteristic case of JC)**.** Is any good endomorphism of the free associative algebra over arbitrary field an automorphism?

3.3.3. Approximation for the automorphism group of affine spaces.

Approximation is the most important tool utilized in this paper. In order to perform it, we have to prove that $\varphi \in \mathrm{Aut}_{\mathrm{Ind}}(\mathrm{Aut}_0(K[x_1, \ldots, x_n])$ preserves the structure of the augmentation subgroup.

The proof method utilized in theorems below works for commutative associative and free associative case. It is a problem of considerable interest to develop similar statements for automorphisms of other associative algebras, such as the commutative Poisson algebra (for which the Aut functor returns the group of polynomial symplectomorphisms); however, the situation there is somewhat more difficult.

Assume that φ is an Ind-automorphism (in either commutative of free associative case) such that it stabilizes point-wise the set T of automorphisms corresponding to the standard diagonal action of the maximal torus (in the next section we will see that this implies that φ also stabilizes every tame automorphism). The following two continuity theorems, for the commutative and the free associative cases, respectively, constitute the foundation of the approximation technique.

Theorem 3.3.6. *Let $\varphi \in \mathrm{Aut}_{\mathrm{Ind}}(\mathrm{Aut}_0(K[x_1, \ldots, x_n]))$ and let $H_N \subset \mathrm{Aut}_0(K[x_1, \ldots, x_n])$ be the subgroup of automorphisms which are identity modulo the ideal $(x_1, \ldots, x_n)^N$, $N > 1$. Then $\varphi(H_N) \subseteq H_N$.*

Theorem 3.3.7. *Let $\varphi \in \mathrm{Aut}_{\mathrm{Ind}}(\mathrm{Aut}_0(K\langle x_1, \ldots, x_n\rangle))$ and let H_N be again the subgroup of automorphisms which are identity modulo the ideal $(x_1, \ldots, x_n)^N$. Then $\varphi(H_N) \subseteq H_N$.*

Corollary 3.3.8. *In both commutative and free associative cases under the assumptions above one has $\varphi = \mathrm{Id}$.*

Proof. Every automorphism can be approximated via the tame ones, i.e., for any ψ and any N there exists a tame automorphism ψ'_N such that $\psi {\psi'_N}^{-1} \in H_N$. □

Therefore, the main point is why $\varphi(H_N) \subseteq H_N$ whenever φ is and Ind-automorphism.

Proof of Theorem 3.3.6. The method of proof is based upon the following useful fact from algebraic geometry.

Lemma 3.3.9. *Let $\varphi : X \to Y$ be a morphism of affine varieties and $A(t) \subset X$ be a curve (or rather, a one-parameter family of points) in X. Assume that $A(t)$ does not tend to infinity as $t \to 0$. Then the image $\varphi A(t)$ under φ also does not tend to infinity as $t \to 0$.*

The proof is straightforward and is left to the reader.

We now put the above fact to use. For $t > 0$ let

$$\hat{A}(t) : \mathbb{A}^n_K \to \mathbb{A}^n_K$$

be a one-parameter family of invertible linear transformations of the affine space preserving the origin. A curve $A(t) \subset \mathrm{Aut}_0(K[x_1, \ldots, x_n])$ of polynomial automorphisms whose points are linear substitutions corresponds to $\hat{A}(t)$. Assume that the ith eigenvalue of $A(t)$ also tends to zero as $t \to 0$ for t^{k_i}, $k_i \in \mathbb{N}$. Such a family will always exist.

Now we assume that the degrees $\{k_i,\ i = 1, \ldots n\}$ of singularity of eigenvalues at zero are such that for every pair (i, j), if $k_i \neq k_j$, then there exists a positive integer m such that

$$\text{either } k_i m \leqslant k_j \text{ or } k_j m \leqslant k_i.$$

The largest number m possessing this property is called the *order* of $A(t)$ at $t = 0$. Since all k_i are positive integers, the order equals the integer part of $k_{\max}/k_{\min}$.

Let $M \in \mathrm{Aut}_0(K[x_1, \ldots, x_n])$ be a polynomial automorphism.

Lemma 3.3.10. *The curve $A(t)MA(t)^{-1}$ has no singularity at zero for any diagonalizable $A(t)$ of order $\leqslant N$ if and only if $M \in \hat{H}_N$, where $\hat{H}_N$ is the subgroup of automorphisms which are homothety modulo the augmentation ideal.*

Proof. The "if" part is elementary, for if $M \in \hat{H}_N$, the action of $A(t)MA(t)^{-1}$ upon any generator x_i (with i fixed)[1] is given by

$$A(t)MA(t)^{-1}(x_i) = \lambda x_i + t^{-k_i} \sum_{l_1+\cdots+l_n=N} a_{l_1 \ldots l_n} t^{k_1 l_1 + \cdots + k_n l_n} x_1^{l_1} \cdots x_n^{l_n} + S_i(t, x_1, \ldots, x_n),$$

where λ is the homothety ratio of (the linear part of) M and S_i is polynomial in $x_1, \ldots, x_n$ of total degree greater than N. Now, for any choice of $l_1, \ldots, l_n$ in the sum, the expression

$$k_1 l_1 + \cdots + k_n l_n - k_i \geqslant k_{\min} \sum l_j - k_i = k_{\min} N - k_i \geqslant 0$$

for every i, so whenever t tends to zero, the coefficient will not tend to infinity. Obviously the same argument applies to higher-degree monomials within S_i.

The other direction is slightly less elementary; assuming that $M \notin \hat{H}_N$, we need to show that there is a curve $A(t)$ such that conjugation of M by it produces a singularity at zero. We distinguish between two cases.

Case 1. The linear part $\bar{M}$ of M is not a scalar matrix. Then after a suitable change of basis (see the footnote), it is not a diagonal matrix and has a nonzero entry in the position (i, j). Consider the diagonal matrix $A(t) = D(t)$ such that on all positions on the main diagonal except jth it has t^{k_i} and on jth position it has t^{k_j}. Then $D(t)\bar{M}D^{-1}(t)$ has (i, j) entry with the coefficient $t^{k_i - k_j}$ and if $k_j > k_i$ it has a singularity at $t = 0$.

Let also $k_i < 2k_j$. Then the nonlinear part of M does not produce singularities and cannot compensate the singularity of the linear part.

[1]Without loss of generality, we may assume that the coordinate functions x_i correspond to the principal axes of $\hat{A}(t)$.

Case 2. The linear part $\bar{M}$ of M is a scalar matrix. Then conjugation cannot produce singularities in the linear part and we as before are interested in the smallest nonlinear term. Let $M \in H_N \backslash H_{N+1}$. Performing a basis change if necessary, we may assume that

$$\varphi(x_1) = \lambda x_1 + \delta x_2^N + S,$$

where S is a sum of monomials of degree $\geqslant N$ with coefficients in K.

Let $A(t) = D(t)$ be a diagonal matrix of the form $(t^{k_1}, t^{k_2}, t^{k_1}, \dots, t^{k_1})$ and let $(N+1) \cdot k_2 > k_1 > N \cdot k_2$. Then in $A^{-1}MA$ the term δx_2^N will be transformed into $\delta x_2^N t^{Nk_2 - k_1}$, and all other terms are multiplied by $t^{lk_2 + sk_1 - k_1}$ with $(l, s) \neq (1, 0)$ and $l, s > 0$. In this case, $lk_2 + sk_1 - k_1 > 0$ and we are done with the proof of Lemma 3.3.10. □

The next lemma is proved by direct computation. Recall that for $m > 1$, the group G_m is defined as the group of all tame automorphisms preserving the mth power of the augmentation ideal.

Lemma 3.3.11.

(a) $[G_m, G_m] \subset H_m$, $m > 2$. *There exist elements* $\varphi \in H_{m+k-1} \backslash H_{m+k}$, $\psi_1 \in G_k$, *and* $\psi_2 \in G_m$ *such that* $\varphi = [\psi_1, \psi_2]$.

(b) $[H_m, H_k] \subset H_{m+k-1}$.

(c) *Let* $\varphi \in G_m \backslash H_m$ *and* $\psi \in H_k \backslash H_{k+1}$, $k > m$. *Then* $[\varphi, \psi] \in H_k \backslash H_{k+1}$.

Proof. (a) Consider elementary automorphisms

$$\psi_1 : x_1 \mapsto x_1 + x_2^k, \quad x_2 \mapsto x_2, \quad x_i \mapsto x_i, \quad i > 2;$$
$$\psi_2 : x_1 \mapsto x_1, \quad x_2 \mapsto x_2 + x_1^m, \quad x_i \mapsto x_i, \quad i > 2.$$

We set $\varphi = [\psi_1, \psi_2] = \psi_1^{-1}\psi_2^{-1}\psi_1\psi_2$. Then

$$\varphi : x_1 \mapsto x_1 - x_2^k + (x_2 - (x_1 - x_2^k)^m)^k,$$
$$x_2 \mapsto x_2 - (x_1 - x_2^k)^m + (x_1 - x_2^k + (x_2 - (x_1 - x_2^k)^m)^k)^m, \quad x_i \mapsto x_i, \quad i > 2.$$

It is easy to see that if either k or m is relatively prime with $\mathrm{char}(K)$, then not all terms of degree $k + m - 1$ vanish. Thus $\varphi \in H_{m+k-1} \backslash H_{m+k}$.

Now assume that $\mathrm{char}(K) \nmid m$; then, obviously, $m - 1$ is relatively prime with $\mathrm{char}(K)$. Consider the mappings

$$\psi_1 : x_1 \mapsto x_1 + x_2^k, \quad x_2 \mapsto x_2, \quad x_i \mapsto x_i, \quad i > 2;$$
$$\psi_2 : x_1 \mapsto x_1, \quad x_2 \mapsto x_2 + x_1^{m-1}x_3, \quad x_i \mapsto x_i, \quad i > 2.$$

Set again $\varphi' = [\psi_1, \psi_2] = \psi_1^{-1}\psi_2^{-1}\psi_1\psi_2$. Then φ' acts as

$$x_1 \mapsto x_1 - x_2^k + \Big(x_2 - (x_1 - x_2^k)^{m-1}x_3\Big)^k = x_1 - k(x_1 - x_2^k)^{m-1}x_2^{k-1}x_3 + S,$$
$$x_2 \mapsto x_2 - (x_1 - x_2^k)^{m-1}x_3 + \Big(x_1 - x_2^k + (x_2 - (x_1 - x_2^k)^{m-1}x_3)^k\Big)^{m-1}x_3,$$
$$x_i \mapsto x_i, \quad i > 2;$$

here S stands for a sum of terms of degree $\geqslant m + k$. Again we see that $\varphi \in H_{m+k-1} \backslash H_{m+k}$.

(b) Let

$$\psi_1 : x_i \mapsto x_i + f_i; \quad \psi_2 : x_i \mapsto x_i + g_i$$

for $i = 1, \dots, n$; here f_i and g_i do not have monomials of degrees less than or equal to m and k, respectively. Then, modulo terms of degree $\geqslant m + k$, we have

$$\psi_1\psi_2 : x_i \mapsto x_i + f_i + g_i + \frac{\partial f_i}{\partial x_j} g_j,$$

so that modulo terms of degree $\geqslant m + k - 1$ we get

$$\psi_1\psi_2 : x_i \mapsto x_i + f_i + g_i, \quad \psi_2\psi_1 : x_i \to x_i + f_i + g_i.$$

Therefore, $[\psi_1, \psi_2] \in H_{m+k-1}$.

(c) If $\varphi(I^m) \subseteq I^m$ and $\psi : (x_1, \ldots, x_n) \mapsto (x_1 + g_1, \ldots, x_n + g_n)$ is such that for some i_0 the polynomial g_{i_0} contains a monomial of total degree k (and all g_i do not contain monomials of total degree less than k), then, by evaluating the composition of automorphisms directly, one sees that the commutator is given by

$$[\varphi, \psi] : (x_1, \ldots, x_n) \mapsto (x_1 + g_1 + S_1, \ldots, x_n + g_n + S_n)$$

with S_i containing no monomials of total degree $<k+1$. Then the image of x_{i_0} is x_{i_0} modulo polynomial of height k. □

Corollary 3.3.12. *Let* $\Psi \in \mathrm{Aut}_{\mathrm{Ind}}(\mathrm{NAut}(K[x_1, \ldots, x_n]))$. *Then* $\Psi(G_n) = G_n$ *and* $\Psi(H_n) = H_n$.

Corollary 3.3.12 together with Proposition 3.4.5 of the next section imply Theorem 3.3.6, for every nice automorphism, by definition, can be approximated by tame ones. Note that in characteristic zero every automorphism is nice (Anick's theorem). □

Proof of Corollary 3.3.12. For simplicity, we set $\operatorname{char} K = 0$, so that NAut coincides with Aut owing to Anick's theorem.

Let φ be an Ind-automorphism which stabilizes point-wise the standard action of the maximal torus.

1. We first note (and give a proof further along the text) that in this case φ also stabilizes point-wise the set of all tame automorphisms.

2. It follows from the singularity trick that $\varphi(\hat{H}_N) \subseteq \hat{H}_N$ (the inverse inclusion is also true due to the invertibility of φ). Namely, if $f = \varphi(g)$ is an automorphism in $\varphi(\hat{H}_N)$ but not in $\hat{H}_N$ then there is a curve $A(t)$ of order $\leqslant N$ such that $A(t) \circ f \circ A(t)^{-1}$ admits a singularity at $t = 0$. But then

$$\varphi^{-1}(A(t) \circ f \circ A(t)^{-1}) = A(t) \circ \varphi^{-1}(f) \circ A(t)^{-1}$$

(this equality holds due to the preservation of tame automorphisms) also admits a singularity at zero, which is a contradiction.

It is a fairly easy exercise to show that for all N we have

$$\varphi(\hat{H}_{N+1} \backslash \hat{H}_N) = \hat{H}_{N+1} \backslash \hat{H}_N.$$

3. Now we demonstrate that $\varphi(\hat{H}_N \backslash H_N) = \hat{H}_N \backslash H_N$ which together with the preceding results will allow us to descend from homothety to identity modulo N.

A. First, let $N > 2$. Assume thatr $g \in \hat{H}_N \backslash H_N$. We take a tame automorphism f which is given by the sum of the identity map and a nonzero term of height two. Consider the automorphism

$$g_f = f \circ g \circ f^{-1}.$$

It is easy to see that $g_f \in \hat{H}_2$: as the linear part of g is given by a scalar matrix not equal to the identity matrix, the degree two component of $g \circ f^{-1}$ is proportional to the homothety ratio $\lambda \neq 1$, therefore the composition with f cannot compensate it.

On the other hand, if $\varphi(g) \in H_N$, i.e., the linear part of $\varphi(g)$ is the identity map, then the degree-two component of $\varphi(g_f) = f \circ \varphi(g) \circ f^{-1}$ (this expression is again due to point-wise preservation of tame automorphisms) is equal to zero, which contradicts the relation $\varphi(\hat{H}_{N+1} \backslash \hat{H}_N) = \hat{H}_{N+1} \backslash \hat{H}_N$.

B. Now let $N = 2$. Assume that $g \in \hat{H}_2 \backslash H_2$ is a nontrivial homothety plus a term of height two. The automorphism g can be approximated by tame automorphisms, in particular there exists a tame automorphism ξ such that $\xi \circ g \in H_3$. The Case A implies that $\varphi(\xi \circ g) = \xi \circ \varphi(g)$ is also in H_3. Since the linear part of g is given by a nontrivial homothety, which means that ξ scales it back to the identity matrix in order to approximate g up to terms of height three, then the left action by ξ^{-1} reverses the scaling, so that the linear part of $\xi^{-1} \circ \xi \circ \varphi(g) = \varphi(g)$ is given by a nontrivial homothety, which implies $\varphi(g) \in \hat{H}_2 \backslash H_2$.

4. Finally, combining all of the results, we get $\varphi(H_N) = H_N$, $N > 1$ as desired. □

3.3.4. Lifting of automorphism groups.

3.3.4.1. Lifting of automorphisms from $\mathrm{Aut}(K[x_1,\dots,x_n])$ *to* $\mathrm{Aut}(K\langle x_1,\dots,x_n\rangle)$.

Definition 3.3.13. In the sequel, an action of the n-dimensional torus $\mathbb{T}^n$ on $\mathbb{K}\langle x_1,\dots,x_n\rangle$ (the number of generators coincides with the dimension of the torus) is said to be *linearizable* if it is conjugate to the standard diagonal action given by

$$(\lambda_1,\dots,\lambda_n):(x_1,\dots,x_n)\mapsto(\lambda_1x_1,\dots,\lambda_nx_n).$$

The following result is a direct free associative analog of a well-known theorem of Białynicki-Birula (see [51, 52]). We will make frequent reference of the classical (commutative) case as well, which appears as Theorem 3.4.1 in the text.

Theorem 3.3.14. *Any effective action of the n-torus on $\mathbb{K}\langle x_1,\dots,x_n\rangle$ is linearizable.*

The proof is somewhat similar to that of Theorem 3.4.1, with a few modifications. We present it in Chapter **??**.

The following proposition is a consequence of Theorem 3.3.14.

Proposition 3.3.15. *Let T^n be the standard torus action on $K[x_1,\dots,x_n]$ and $\widehat{T}^n$ be its lifting to an action on the free associative algebra $K\langle x_1,\dots,x_n\rangle$. Then $\widehat{T}^n$ is also given by the standard torus action.*

Consider the roots $\widehat{x_i}$ of this action. They are liftings of the coordinates x_i. We must prove that they generate the whole associative algebra.

Due to the reducibility of this action, all elements are products of eigenvalues of this action. Hence it suffices to prove that eigenvalues of this action can be presented as a linear combination of this action. This can be done similarly to [51]. Note that all propositions of the previous section hold for the free associative algebra. Proof of Theorem 3.3.7 is similar. Hence we have the following theorem.

Theorem 3.3.16. *Any* Ind*-scheme automorphism φ of* $\mathrm{Aut}(K\langle x_1,\dots,x_n\rangle)$ *for $n\geqslant 3$ is inner, i.e., is a conjugation by some automorphism.*

Therefore, we see that the group lifting (in the sense of isomorphism induced by the natural abelianization) implies an analog of Theorem 3.3.6.

This also implies that any automorphism group lifting, if exists, satisfies the approximation properties.

Proposition 3.3.17. *Assume that*

$$\Psi:\mathrm{Aut}(K[x_1,\dots,x_n])\to\mathrm{Aut}(K\langle z_1,\dots,z_n\rangle)$$

is a group homomorphism such that its composition with the natural map

$$\mathrm{Aut}(K\langle z_1,\dots,z_n\rangle)\to\mathrm{Aut}(K[x_1,\dots,x_n])$$

(induced by the projection $K\langle z_1,\dots,z_n\rangle\to K[x_1,\dots,x_n]$) is the identity map.

1. *The coordinate change Ψ provides a correspondence between the standard torus actions $x_i\mapsto\lambda_ix_i$ and $z_i\mapsto\lambda_iz_i$.*
2. *The images of elementary automorphisms*

$$x_j\mapsto x_j,\quad j\neq i,\quad x_i\mapsto x_i+f(x_1,\dots,\widehat{x_i},\dots,x_n)$$

are elementary automorphisms of the form

$$z_j\mapsto z_j,\quad j\neq i,\quad z_i\mapsto z_i+f(z_1,\dots,\widehat{z_i},\dots,z_n).$$

Hence the image of a tame automorphism is a tame automorphism.

3. *$\psi(H_n)=G_n$. Hence ψ induces a map between the completion of the groups of* $\mathrm{Aut}(K[x_1,\dots,x_n])$ *and* $\mathrm{Aut}(K\langle z_1,\dots,z_n\rangle)$ *with respect to the augmentation subgroup structure.*

Proof of Theorem 3.1.14. Any automorphism (including wild automorphisms such as the Nagata example) can be approximated by a product of elementary automorphisms with respect to augmentation topology. In the case of the Nagata automorphism corresponding to

$$\mathrm{Aut}(K\langle x_1, \ldots, x_n\rangle),$$

all such elementary automorphisms fix all coordinates except x_1 and x_2. Because of items 2 and 3 of Proposition 3.3.17, the lifted automorphism would be an automorphism induced by an automorphism of $K\langle x_1, x_2, x_3\rangle$ fixing x_3. However, it is impossible to lift the Nagata automorphism to such an automorphism due to the main result of [38]. Theorem 3.1.14 is proved. □

3.4. Automorphisms of the Polynomial Algebra and the Bodnarchuk–Rips Approach

Assume that

$$\Psi \in \mathrm{Aut}(\mathrm{Aut}(K[x_1, \ldots, x_n])) \quad \text{or} \quad \Psi \in \mathrm{Aut}(\mathrm{TAut}(K[x_1, \ldots, x_n]))$$
$$\Psi \in \mathrm{Aut}(\mathrm{TAut}_0(K[x_1, \ldots, x_n])) \quad \text{or} \quad \Psi \in \mathrm{Aut}(\mathrm{Aut}_0(K[x_1, \ldots, x_n])).$$

3.4.1. Reduction to the case where Ψ is identical on SL_n. We follow [133] and [55] using the classical Białynicki-Birula theorem (see [51, 52]).

Theorem 3.4.1 (Białynicki-Birula). *Any effective action of torus $\mathbb{T}^n$ on $\mathbb{C}^n$ is linearizable* (cf. Definition 3.3.13).

Remark 3.4.2. An effective action of $\mathbb{T}^{n-1}$ on $\mathbb{C}^n$ is linearizable (see [51,52]). There is a conjecture whether any action of $\mathbb{T}^{n-2}$ on $\mathbb{C}^n$ is linearizable, established for $n = 3$. For codimension >2, there are positive-characteristic counterexamples (see [18]).

Remark 3.4.3. By considering periodic elements in $\mathbb{T}$, Kraft and Stampfli proved (see [133]) that an effective action T possesses the following property: if $\Psi \in \mathrm{Aut}(\mathrm{Aut})$ is a group automorphism, then the image of T (as a subgroup of Aut) under Ψ is an algebraic group. In fact, their proof is also applicable for the free associative algebra case. We use this result.

Returning to the case of automorphisms $\varphi \in \mathrm{Aut}_{\mathrm{Ind}}\,\mathrm{Aut}$ preserving the Ind-group structure, we now consider the standard action $x_i \mapsto \lambda_i x_i$ of the n-dimensional torus $\mathbb{T} \leftrightarrow T^n \subset \mathrm{Aut}(\mathbb{C}[x_1, \ldots, x_n])$ on the affine space $\mathbb{C}^n$. Let H be the image of T^n under φ. Then by Theorem 3.4.1, H is conjugate to the standard torus T^n via some automorphism ψ. Composing φ with this conjugation, we come to the case where φ is the identity on the maximal torus. Then we have the following assertion.

Corollary 3.4.4. *Without loss of generality, it suffices to prove Theorem 3.1.6 for the case where* $\varphi|_{\mathbb{T}} = \mathrm{Id}$.

Now we are in the situation when φ preserves all linear mappings $x_i \mapsto \lambda_i x_i$. We must prove that it is the identity.

Proposition 3.4.5 (E. Rips, private communication). *Let $n > 2$. Assume that φ preserves the standard torus action on the commutative polynomial algebra. Then φ preserves all elementary transformations.*

Corollary 3.4.6. *Let φ satisfy the conditions of Proposition* 3.4.5. *Then φ preserves all tame automorphisms.*

Proof of Proposition 3.4.5. We state a few elementary lemmas.

Lemma 3.4.7. *Consider the diagonal action $T^1 \subset T^n$ given by automorphisms*

$$\alpha : x_i \mapsto \alpha_i x_i, \quad \beta : x_i \mapsto \beta_i x_i.$$

Let $\psi : x_i \mapsto \sum\limits_{i,J} a_{iJ} x^J$, $i = 1, \ldots, n$, *where* $J = (j_1, \ldots, j_n)$ *is the multi-index,* $x^J = x^{j_1} \cdots x^{j_n}$. *Then*

$$\alpha \circ \psi \circ \beta : x_i \mapsto \sum_{i,J} \alpha_i a_{iJ} x^J \beta^J,$$

In particular,

$$\alpha \circ \psi \circ \alpha^{-1} : x_i \mapsto \sum_{i,J} \alpha_i a_{iJ} x^J \alpha^{-J}.$$

Applying Lemma 3.4.7 and comparing the coefficients, we obtain the following assertion.

Lemma 3.4.8. *Consider the diagonal* T^1 *action* $x_i \mapsto \lambda x_i$. *Then the set of automorphisms commuting with this action is exactly the set of linear automorphisms.*

Similarly (using Lemma 3.4.7), we obtain Lemmas 3.4.9, 3.4.12, and 3.4.13.

Lemma 3.4.9.

(a) *Consider the following* T^2 *action*:

$$x_1 \mapsto \lambda\delta x_1, \quad x_2 \mapsto \lambda x_2, \quad x_3 \mapsto \delta x_3, \quad x_i \mapsto \lambda x_i, \quad i > 3.$$

Then the set S *of automorphisms commuting with this action is generated by the following automorphisms*:

$$x_1 \mapsto x_1 + \beta x_2 x_3, \quad x_i \mapsto \varepsilon_i x_i, \quad i > 1, \quad (\beta, \varepsilon_i \in K).$$

(b) *Consider the following* T^{n-1} *action*:

$$x_1 \mapsto \lambda^I x_1, \quad x_j \mapsto \lambda_j x_j, \quad j > 1, \quad \lambda^I = \lambda_2^{i_2} \cdots \lambda_n^{i_n}.$$

Then the set S *of automorphisms commuting with this action is generated by the following automorphisms*:

$$x_1 \mapsto x_1 + \beta \prod_{j=2}^{n} x_j^{i_j}, \quad \beta \in K.$$

Remark 3.4.10. A similar statement for the free associative case is true, but one has to consider the set $\hat{S}$ of automorphisms $x_1 \mapsto x_1 + h$, $x_i \mapsto \varepsilon_i x_i$, $i > 1$ ($\varepsilon \in K$, and the polynomial $h \in K\langle x_2, \ldots, x_n\rangle$ has total degree J; in the free associative case it is not just monomial anymore).

Corollary 3.4.11. *Assume that* $\varphi \in \mathrm{Aut}(\mathrm{TAut}(K[x_1, \ldots, x_n]))$ *stabilizes all elements from* $\mathbb{T}$. *Then* $\varphi(S) = S$.

Lemma 3.4.12. *Consider the following* T^1 *action*:

$$x_1 \mapsto \lambda^2 x_1, \quad x_i \mapsto \lambda x_i, \quad i > 1.$$

Then the set S *of automorphisms commuting with this action is generated by the following automorphisms:*

$$x_1 \mapsto x_1 + \beta x_2^2, \quad x_i \mapsto \lambda_i x_i, \quad i > 2, \quad \beta, \lambda_i \in K.$$

Lemma 3.4.13. *Consider the set* S *defined in the previous lemma. Then* $[S, S] = \{uvu^{-1}v^{-1}\}$ *consists of the following automorphisms*:

$$x_1 \mapsto x_1 + \beta x_2 x_3, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3, \quad \beta \in K.$$

Lemma 3.4.14. *Let* $n \geqslant 3$. *Consider the following set of automorphisms:*

$$\psi_i : x_i \mapsto x_i + \beta_i x_{i+1} x_{i+2}, \quad \beta_i \neq 0, \quad x_k \mapsto x_k, \quad k \neq i,$$

for $i = 1, \ldots, n-1$. *(Numeration is cyclic, so, for example,* $x_{n+1} = x_1$.*) Let* $\beta_i \neq 0$ *for all* i. *Then each of* ψ_i *can be uniquely simultaneously conjugated by a torus action to*

$$\psi_i' : x_i \mapsto x_i + x_{i+1} x_{i+2}, \quad x_k \mapsto x_k, \quad k \neq i,$$

for $i = 1, \ldots, n$.

Proof. Let $\alpha : x_i \mapsto \alpha_i x_i$. Then by Lemma 3.4.7 we obtain

$$\alpha \circ \psi_i \circ \alpha^{-1} : x_i \mapsto x_i + \beta_i x_{i+1} x_{i+2} \alpha_{i+1}^{-1} \alpha_{i+2}^{-1} \alpha_i, \quad \alpha \circ \psi_i \circ \alpha^{-1} : x_k \mapsto x_k$$

for $k \neq i$. Comparing the coefficients of the quadratic terms, we see that it suffices to solve the system

$$\beta_i \alpha_{i+1}^{-1} \alpha_{i+2}^{-1} \alpha_i = 1, \ i = 1, \dots, n-1.$$

Since $\beta_i \neq 0$ for all i, this system has a unique solution. □

Remark 3.4.15. In the case of a free associative algebra, one must consider $\beta x_2 x_3 + \gamma x_3 x_2$ instead of $\beta x_2 x_3$.

Proposition 3.4.5 follows from Lemmas 3.4.8, 3.4.9, 3.4.12, 3.4.13, 3.4.14, and 3.4.16 (see below). Note that we have proved an analog of Theorem 3.1.6 for tame automorphisms. □

3.4.2. The Rips lemma.

Lemma 3.4.16 (E. Rips)**.** *Let* $\mathrm{char}(K) \neq 2$, $|K| = \infty$. *Linear transformations and* ψ_i' *defined in Lemma 3.4.14 generate the whole tame automorphism group of* $K[x_1, \dots, x_n]$.

Proof of Lemma 3.4.16. Let G be the group generated by elementary transformations as in Lemma 3.4.14. We must prove that is isomorphic to the tame automorphism subgroup fixing the augmentation ideal. We need some preliminaries. □

Lemma 3.4.17. *Linear transformations of* K^3 *and*

$$\psi : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy$$

generate all mappings of the form

$$\phi_m^b(x, y, z) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + bx^m, \quad b \in K.$$

Proof of Lemma 3.4.17. We proceed by induction. Assume that Suppose we have an automorphism

$$\phi_{m-1}^b(x, y, z) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + bx^{m-1}.$$

Conjugating by the linear transformation ($z \mapsto y$, $y \mapsto z$, $x \mapsto x$), we obtain the automorphism

$$\phi_{m-1}^b(x, z, y) : x \mapsto x, \ y \mapsto y + bx^{m-1}, \ z \mapsto z.$$

Composing this on the right by ψ, we get the automorphism

$$\varphi(x, y, z) : x \mapsto x, \ y \mapsto y + bx^{m-1}, \ z \mapsto z + yx + x^m.$$

Note that

$$\phi_{m-1}(x, y, z)^{-1} \circ \varphi(x, y, z) : x \mapsto x, \ y \mapsto y, \ z \mapsto z + xy + bx^m.$$

Now we see that

$$\psi^{-1} \phi_{m-1}(x, y, z)^{-1} \circ \varphi(x, y, z) = \phi_m^b.$$ □

Corollary 3.4.18. *Let* $\mathrm{char}(K) \nmid n$ (*in particular,* $\mathrm{char}(K) \neq 0$) *and* $|K| = \infty$. *Then* G *contains all the transformations*

$$z \mapsto z + bx^k y^l, \ y \mapsto y, \ x \mapsto x$$

such that $k + l = n$.

Proof. For any invertible linear transformation

$$\varphi : x \mapsto a_{11}x + a_{12}y, \ y \mapsto a_{21}x + a_{22}y, \ z \mapsto z; \quad a_{ij} \in K,$$

we have

$$\varphi^{-1} \phi_m^b \varphi : x \mapsto x, \ y \mapsto y, \ z \mapsto z + b(a_{11}x + a_{12}y)^m.$$

Note that sums of such expressions contain all the terms of the form $bx^k y^l$. □

3.4.3. Generators of the tame automorphism group.

Theorem 3.4.19. *If* $\mathrm{char}(K) \neq 2$ *and* $|K| = \infty$*, then linear transformations and*

$$\psi : x \mapsto x,\ y \mapsto y,\ z \mapsto z + xy$$

generate all mappings of the form

$$\alpha_m^b(x, y, z) : x \mapsto x,\ y \mapsto y,\ z \mapsto z + byx^m, \quad b \in K.$$

Proof of Theorem 3.4.19. Observe that

$$\alpha = \beta \circ \phi_m^b(x, z, y) : x \mapsto x + by^m,\ y \mapsto y + x + by^m,\ z \mapsto z,$$

where $\beta : x \mapsto x$, $y \mapsto x + y$, $z \mapsto z$. Then

$$\gamma = \alpha^{-1}\psi\alpha : x \mapsto x,\ y \mapsto y,\ z \mapsto z + xy + 2bxy^m + by^{2m}.$$

Composing with ψ^{-1} and ϕ_{2m}^{2b} we obtain the required fact:

$$\alpha_m^{2b}(x, y, z) : x \mapsto x,\ y \mapsto y,\ z \mapsto z + 2byx^m, \quad b \in K. \qquad \square$$

Corollary 3.4.20. *Let* $\mathrm{char}(K) \nmid n$ *and* $|K| = \infty$*. Then* G *contains all transformations of the form*

$$z \mapsto z + bx^k y^l,\ y \mapsto y,\ x \mapsto x$$

such that $k = n + 1$*.*

The proof is similar to the proof of Corollary 3.4.18. Note that either n or $n + 1$ is not a multiple of $\mathrm{char}(K)$ so we have the following assertion.

Lemma 3.4.21. *If* $\mathrm{char}(K) \neq 2$*, then linear transformations and*

$$\psi : x \mapsto x,\ y \mapsto y,\ z \mapsto z + xy$$

generate all mappings of the form

$$\alpha_P : x \mapsto x,\ y \mapsto y,\ z \mapsto z + P(x, y), \quad P(x, y) \in K[x, y].$$

We have proved Lemma 3.4.16 for the three variable case.

In order to treat the case $n \geqslant 4$, we need the following lemma.

Lemma 3.4.22. *Let* $M(\vec{x}) = a \prod x_i^{k_i}$*,* $a \in K$*,* $|K| = \infty$*,* $\mathrm{char}(K) \nmid k_i$ *for at least one of* k_i*'s. Consider the linear transformations*

$$f : x_i \mapsto y_i = \sum a_{ij} x_j, \quad \det(a_{ij}) \neq 0,$$

and monomials $M_f = M(\vec{y})$*. Then the linear span of* M_f *for various* f *contains all homogenous polynomials of degree* $k = \sum k_i$ *in* $K[x_1, \dots, x_n]$*.*

Proof. Lemma 3.4.22 is a direct consequence of the following fact. Let S be a homogenous subspace of $K[x_1, \dots, x_n]$ invariant with respect to GL_n of degree m. Then $S = S_{m/p^k}^{p^k}$, $p = \mathrm{char}(K)$, S_l is the space of all polynomials of degree l. $\square$

Lemma 3.4.16 follows from Lemma 3.4.22 in a similar way as in the proofs of Corollaries 3.4.18 and 3.4.20.

3.4.4. Aut(TAut) **in the general case.** Now we consider the case where $\mathrm{char}(K)$ is arbitrary, i.e., the remaining case $\mathrm{char}(K) = 2$. Still $|K| = \infty$. Although we are unable to prove an analog of Proposition 3.4.5, we can still play on the relations.

Let

$$M = a \prod_{i=1}^{n-1} x_i^{k_i}$$

be a monomial, $a \in K$. For polynomial $P(x, y) \in K[x, y]$ we define the elementary automorphism

$$\psi_P : x_i \mapsto x_i, \quad i = 1, \dots, n-1, \quad x_n \mapsto x_n + P(x_1, \dots, x_{n-1}).$$

We have $P = \sum M_j$ and ψ_P can be naturally decomposed as the product of commuting ψ_{M_j}. Let $\Psi \in \mathrm{Aut}(\mathrm{TAut}(K[x, y, z]))$ stabilize linear mappings and ϕ (the automorphism ϕ was defined in Lemma 3.4.17). Then, according to Corollary 3.4.11, $\Psi(\psi_P) = \prod \Psi(\psi_{M_j})$. If $M = ax^n$, then due to Lemma 3.4.17, we have $\Psi(\psi_M) = \psi_M$. We must prove the same for other type of monomials.

Lemma 3.4.23. *Let M be a monomial. Then $\Psi(\psi_M) = \psi_M$.*

Proof. Let $M = a \prod_{i=1}^{n-1} x_i^{k_i}$. Consider the automorphism

$$\alpha : x_i \mapsto x_i + x_1, \quad i = 2, \dots, n-1; \quad x_1 \mapsto x_1, \quad x_n \mapsto x_n.$$

Then

$$\alpha^{-1} \psi_M \alpha = \psi_{x_1^{k_1} \prod\limits_{i=2}^{n-1} (x_i + x_1)^{k_i}} = \psi_Q \psi_{a x_1^{\sum_{i=2}^{n-1} k_i}}.$$

Here the polynomial

$$Q = x_1^{k_1} \left(\prod_{i=2}^{n-1} (x_i + x_1)^{k_i} - a x_1^{\sum k_i} \right).$$

It has the form $Q = \sum\limits_{i=2}^{n-1} N_i$, where N_i are monomials such that none of them is proportional to a power of x_1.

According to Corollary 3.4.11, $\Psi(\psi_M) = \psi_{bM}$ for some $b \in K$. We need only to prove that $b = 1$. Assume the contrary, i.e., $b \neq 1$. Then

$$\Psi(\alpha^{-1} \psi_M \alpha) = \left(\prod_{[N_i, x_1] \neq 0} \Psi(\psi_{N_i}) \right) \circ \Psi(\psi_{a x_1^{\sum_{i=2}^{n-1} k_i}}) = \left(\prod_{[N_i, x_1] \neq 0} \psi_{b_i N_i} \right) \circ \psi_{a x_1^{\sum_{i=2}^{n-1} k_i}}$$

for some $b_i \in K$. On the other hand,

$$\Psi(\alpha^{-1} \psi_M \alpha) = \alpha^{-1} \Psi(\psi_M) \alpha = \alpha^{-1} \psi_{bM} \alpha = \left(\prod_{[N_i, x_1] \neq 0} \psi_{b N_i} \right) \circ \psi_{a x_1^{\sum_{i=2}^{n-1} k_i}}.$$

Comparing the factors $\psi_{a x_1^{\sum_{i=2}^{n-1} k_i}}$ and $\psi_{a x_1^{\sum_{i=2}^{n-1} k_i}}$ in the last two products we get $b = 1$. Lemma 3.4.23 and hence Proposition 3.4.5 are proved. □

3.5. The Bodnarchuk–Rips Approach to Automorphisms of $\mathrm{TAut}(K\langle x_1, \dots, x_n \rangle)$ $(n > 2)$

Now we consider the free associative case. We treat the case $n > 3$ on the group-theoretic level and the case $n = 3$ on the Ind-scheme level. Note that if $n = 2$, then

$$\mathrm{Aut}_0(K[x, y]) = \mathrm{TAut}_0(K[x, y]) \simeq \mathrm{TAut}_0(K\langle x, y \rangle) = \mathrm{Aut}_0(K\langle x, y \rangle)$$

and description of automorphism group of such objects is known due to J. Déserti.

3.5.1. The automorphisms of the tame automorphism group of $K\langle x_1,\dots,x_n\rangle$, $n\geqslant 4$.

Proposition 3.5.1 (E. Rips, private communication)**.** *Let $n>3$ and let φ preserve the standard torus action on the free associative algebra $K\langle x_1,\dots,x_n\rangle$. Then φ preserves all elementary transformations.*

Corollary 3.5.2. *Let φ satisfy the conditions of Proposition 3.5.1. Then φ preserves all tame automorphisms.*

For free associative algebras, we note that any automorphism preserving the torus action preserves also the symmetric

$$x_1\mapsto x_1+\beta(x_2x_3+x_3x_2),\quad x_i\mapsto x_i,\quad i>1,$$

and the skew symmetric

$$x_1\mapsto x_1+\beta(x_2x_3-x_3x_2),\quad x_i\mapsto x_i,\quad i>1,$$

elementary automorphisms. The first property follows from Lemma 3.4.12. The second one follows from the fact that skew symmetric automorphisms commute with automorphisms of the following type

$$x_2\mapsto x_2+x_3^2,\quad x_i\mapsto x_i,\quad i\neq 2,$$

and this property distinguishes them from elementary automorphisms of the form

$$x_1\mapsto x_1+\beta x_2x_3+\gamma x_3x_2,\quad x_i\mapsto x_i,\quad i>1.$$

Theorem 3.1.7 follows from the fact that the forms $\beta x_2x_3+\gamma x_3x_2$ corresponding to general bilinear multiplication

$$*_{\beta,\gamma}:(x_2,\ x_3)\to\beta x_2x_3+\gamma x_3x_2$$

lead to associative multiplication if and only if $\beta=0$ or $\gamma=0$; the approximation also applies (see Sec. 3.3.3).

First, assume that $n=4$ and consider $K\langle x,y,z,t\rangle$.

Proposition 3.5.3. *The group G containing all linear transformations and mappings*

$$x\mapsto x,\ y\mapsto y,\ z\mapsto z+xy,\ t\mapsto t$$

contains also all transformations of the form

$$x\mapsto x,\ y\mapsto y,\ z\mapsto z+P(x,y),\ t\mapsto t.$$

Proof. It suffices to prove that G contains all transformations of the form

$$x\mapsto x,\ y\mapsto y,\ z\mapsto z+aM,\ t\mapsto t,\quad a\in K,$$

where M is a monomial.

Step 1. Let

$$M=a\prod_{i=1}^m x^{k_i}y^{l_i}\quad\text{or}\quad M=a\prod_{i=1}^m y^{l_0}x^{k_i}y^{l_i}\quad\text{or}\quad M=a\prod_{i=1}^m x^{k_i}y^{l_i}\quad\text{or}\quad M=a\prod_{i=1}^m x^{k_i}y^{l_i}x^{k_{m+1}}.$$

Define the height $H(M)$ of M as the number of segments involved of a specific generator—such as x^k—in the word M. For example,

$$H\left(a\prod_{i=1}^m x^{k_i}y^{l_i}x^{k_{m+1}}\right)=2m+1.$$

Using induction on $H(M)$, one can reduce to the case where $M=yx^k$. Let $M=M'x^k$ such that $H(M')<H(M)$. (The case where $M=M'y^l$ is similar.) Let

$$\phi:x\to x,\ y\to y,\ z\to z+M',\ t\to t;\quad \alpha:x\to x,\ y\to y,\ z\to z,\ t\to t+zx^k.$$

Then

$$\phi^{-1}\circ\alpha\circ\phi:x\to x,\ y\to y,\ z\to z,\ t\to t-M+zx^k.$$

The automorphism $\phi^{-1} \circ \alpha \circ \phi$ is the composition of automorphisms

$$\beta : x \to x,\ y \to y,\ z \to z,\ t \to t - M \quad \text{and} \quad \gamma : x \to x,\ y \to y,\ z \to z,\ t \to t + zx^k.$$

Observe that β is conjugate to the automorphism

$$\beta' : x \to x,\ y \to y,\ z \to z - M,\ t \to t$$

by a linear automorphism

$$x \to x,\ y \to y,\ z \to t,\ t \to z.$$

Similarly, γ is conjugate to the automorphism

$$\gamma' : x \to x,\ y \to y,\ z \to z + yx^k,\ t \to t.$$

We have thus reduced to the case where $M = x^k$ or $M = yx^k$.

Step 2. Consider the automorphisms

$$\alpha : x \to x,\ y \to y + x^k,\ z \to z,\ t \to t \quad \text{and} \quad \beta : x \to x,\ y \to y,\ z \to z,\ t \to t + azy.$$

Then

$$\alpha^{-1} \circ \beta \circ \alpha : x \to x,\ y \to y,\ z \to z,\ t \to t + azx^k + azy.$$

It is a composition of the automorphism

$$\gamma : x \to x,\ y \to y,\ z \to z,\ t \to t + azx^k,$$

which is conjugate to the needed automorphism

$$\gamma' : x \to x,\ y \to y,\ z \to z + yx^k,\ t \to t$$

and an automorphism

$$\delta : x \to x,\ y \to y,\ z \to z,\ t \to t + azy,$$

which is conjugate to the automorphism

$$\delta' : x \to x,\ y \to y,\ z \to z + axy,\ t \to t$$

and then to the automorphism

$$\delta'' : x \to x,\ y \to y,\ z \to z + xy,\ t \to t$$

(using similarities). We have reduced the problem to proving the statement

$$G \ni \psi_M, \quad M = x^k \quad \text{for all } k.$$

Step 3. Obtain the automorphism

$$x \to x,\ y \to y + x^n,\ z \to z,\ t \to t.$$

This problem is similar to the commutative case of $K[x_1, \dots, x_n]$ (cf. Sec. 3.4).

Proposition 3.5.3 is proved. □

Returning to the general case $n \geqslant 4$, let us formulate Remark 3.4.10 as follows.

Lemma 3.5.4. *Consider the following* T^{n-1} *action*:

$$x_1 \to \lambda^I x_1, \quad x_j \to \lambda_j x_j, \quad j > 1; \quad \lambda^I = \lambda_2^{i_2} \cdots \lambda_n^{i_n}.$$

Then the set S *of automorphisms commuting with this action is generated by the following automorphisms:*

$$x_1 \to x_1 + H, \quad x_i \to x_i; \quad i > 1,$$

where H *is any homogenous polynomial of total degree* $i_2 + \cdots + i_n$.

Proposition 3.5.3 and Lemma 3.5.4 imply the following assertion.

Corollary 3.5.5. *Let* $\Psi \in \mathrm{Aut}(\mathrm{TAut}_0(K\langle x_1,\ldots,x_n\rangle))$ *stabilize all elements of torus and linear automorphisms*

$$\phi_P : x_n \to x_n + P(x_1,\ldots,x_{n-1}),\quad x_i \to x_i,\quad i=1,\ldots,n-1.$$

Let $P = \sum_I P_I$, *where* P_I *is the homogenous component of* P *of multi-degree* I. *Then the following assertions hold*:

9(a) $\Psi(\phi_P) : x_n \to x_n + P^\Psi(x_1,\ldots,x_{n-1})$, $x_i \to x_i$, $i=1,\ldots,n-1$.
9(b) $P^\Psi = \sum_I P_I^\Psi$; *here* P_I^Ψ *is homogenous of multi-degree* I.
9(c) *If* I *has positive degree with respect to one or two variables, then* $P_I^\Psi = P_I$.

Let $\Psi \in \mathrm{Aut}(\mathrm{TAut}_0(K\langle x_1,\ldots,x_n\rangle))$ stabilize all elements of torus and linear automorphisms,

$$\phi : x_n \to x_n + P(x_1,\ldots,x_{n-1}),\quad x_i \to x_i,\quad i=1,\ldots,n-1.$$

Let

$$\varphi_Q : x_1 \to x_1,\ x_2 \to x_2,\ x_i \to x_i + Q_i(x_1,x_2),\quad i=3,\ldots,n-1,\quad x_n \to x_n;$$

$Q=(Q_3,\ldots,Q_{n-1})$. Then $\Psi(\varphi_Q)=\varphi_Q$ by Proposition 3.5.3.

Lemma 3.5.6.

(a) $\varphi_Q^{-1}\circ\phi_P\circ\varphi_Q = \phi_{P_Q}$, *where*

$$P_Q(x_1,\ldots,x_{n-1}) = P\Big(x_1,x_2,x_3+Q_3(x_1,x_2),\ldots,x_{n-1}+Q_{n-1}(x_1,x_2)\Big).$$

(b) *Let* $P_Q = P_Q^{(1)} + P_Q^{(2)}$ *and* $P_Q^{(1)}$ *consist of all terms containing one of the variables* $x_3,\ldots,x_{n-1}$, *and let* $P_Q^{(1)}$ *consist of all terms containing just* x_1 *and* x_2. *Then*

$$P_Q^\Psi = P_Q^\Psi = P_Q^{(1)\Psi} + P_Q^{(2)\Psi} = P_Q^{(1)\Psi} + P_Q^{(2)}.$$

Lemma 3.5.7. *If* $P_Q^{(2)} = R_Q^{(2)}$ *for all* Q, *then* $P=R$.

Proof. It suffices to prove that if $P\neq 0$, then $P_Q^{(2)}\neq 0$ for appropriate $Q=(Q_3,\ldots,Q_{n-1})$. Let $m=\deg(P)$, $Q_i = x_1^{2^{i+1}m}x_2^{2^{i+1}m}$. Let $\hat P$ be the highest-degree component of P, then $\hat P(x_1,x_2,Q_3,\ldots,Q_{n-1})$ is the highest-degree component of $P_Q^{(2)}$. It suffices to prove that

$$\hat P(x_1,x_2,Q_3,\ldots,Q_{n-1})\neq 0.$$

Let $x_1 \prec x_2 \prec x_2 \prec \cdots \prec x_{n-1}$ be the standard lexicographic order. Consider the lexicographically minimal term M of $\hat P$. It is easy to see that the term $M\big|_{Q_i\to x_i}$, $i=3,n-1$, cannot cancel with any other term $N\big|_{Q_i\to x_i}$, $i=3,n-1$, of $\hat P(x_1,x_2,Q_3,\ldots,Q_{n-1})$. Therefore, $\hat P(x_1,x_2,Q_3,\ldots,Q_{n-1})\neq 0$. □

Lemmas 3.5.6 and 3.5.7 imply the following assertion.

Corollary 3.5.8. *Let* $\Psi \in \mathrm{Aut}(\mathrm{TAut}_0(K\langle x_1,\ldots,x_n\rangle))$ *stabilize all elements of torus and linear automorphisms. Then* $P^\Psi = P$, *and* Ψ *stabilizes all elementary automorphisms and, therefore, the entire group* $\mathrm{TAut}_0(K\langle x_1,\ldots,x_n\rangle)$.

Proposition 3.5.9. *Let* $n\geqslant 4$ *and let* $\Psi \in \mathrm{Aut}(\mathrm{TAut}_0(K\langle x_1,\ldots,x_n\rangle))$ *stabilize all elements of torus and linear automorphisms. Then either* $\Psi=\mathrm{Id}$ *or* Ψ *acts as conjugation by the mirror anti-automorphism.*

Let $n\geqslant 4$. Let $\Psi \in \mathrm{Aut}(\mathrm{TAut}_0(K\langle x_1,\ldots,x_n\rangle))$ stabilize all elements of torus and linear automorphisms. Denote by EL an elementary automorphism

$$EL : x_1 \to x_1,\ \ldots,\ x_{n-1}\to x_{n-1},\ x_n \to x_n + x_1x_2$$

(all other elementary automorphisms of this form, i.e., $x_k \to x_k + x_i x_j$, $x_l \to x_l$ for $l \neq k$ and $k \neq i$, $k \neq j$, $i \neq j$, are conjugate to one another by permutations of generators).

We must prove that $\Psi(EL) = EL$ or $\Psi(EL) : x_i \to x_i$, $i = 1, \ldots, x_{n-1}$, $x_n \to x_n + x_2 x_1$. The latter corresponds to Ψ being the conjugation with the mirror anti-automorphism of $K\langle x_1, \ldots, x_n\rangle$.

For some $a, b \in K$, we define $x *_{a,b} y = axy + byx$. Then, in any of the above two cases,

$$\Psi(EL) : x_i \to x_i; \quad i = 1, \ldots, x_{n-1}, \quad x_n \to x_n + x_1 *_{a,b} x_2$$

for some a and b.

The following lemma is elementary.

Lemma 3.5.10. *The operation $* = *_{a,b}$ is associative if and only if $ab = 0$.*

The associator of x, y, and z is defined as follows:

$$\{x, y, z\}_* \equiv (x * y) * z - x * (y * z) = ab(zx - xz)y + aby(xz - zx) = ab[y, [x, z]].$$

Now we are ready to prove Proposition 3.5.9. For simplicity, we treat only the case $n = 4$, the general case is considered similarly. Consider the automorphisms

$$\begin{aligned} \alpha &: x \to x, \quad y \to y, \quad z \to z + xy, \quad t \to t, \\ \beta &: x \to x, \quad y \to y, \quad z \to z, \quad t \to t + xz, \\ h &: x \to x, \quad y \to y, \quad z \to z, \quad t \to t - xz; \end{aligned}$$

here $h = \beta^{-1}$. Then

$$\gamma = h\alpha^{-1}\beta\alpha = [\beta, \alpha] : x \to x,\ y \to y,\ z \to z,\ t \to t - x^2 y.$$

Note that α is conjugate to β via a generator permutation

$$\kappa : x \to x,\ y \to z,\ z \to t,\ t \to y, \quad \kappa \circ \alpha \circ \kappa^{-1} = \beta$$

and

$$\Psi(\gamma) : x \to x,\ y \to y,\ z \to z,\ t \to t - x * (x * y).$$

Let

$$\begin{aligned} \delta &: x \to x,\ y \to y,\ z \to z + x^2,\ t \to t, \\ \epsilon &: x \to x,\ y \to y,\ z \to z,\ t \to t + zy. \end{aligned}$$

Let $\gamma' = \epsilon^{-1}\delta^{-1}\epsilon\delta$. Then

$$\gamma' : x \to x,\ y \to y,\ z \to z,\ t \to t - x^2 y.$$

On the other hand we have

$$\varepsilon = \Psi(\epsilon^{-1}\delta^{-1}\epsilon\delta) : x \to x,\ y \to y,\ z \to z,\ t \to t - (x^2) * y.$$

We also have $\gamma = \gamma'$. Equality $\Psi(\gamma) = \Psi(\gamma')$ is equivalent to the equality $x * (x * y) = x^2 * y$. This implies $x * y = xy$ and we are done.

3.5.2. The group $\mathrm{Aut}_{\mathrm{Ind}}(\mathrm{TAut}(K\langle x, y, z\rangle))$. This is the most technically loaded part of the present study. At the moment we are unable to accomplish the objective of describing the whole group $\mathrm{Aut}\,\mathrm{TAut}(K\langle x, y, z\rangle)$. In this section we will determine only its subgroup $\mathrm{Aut}_{\mathrm{Ind}}\,\mathrm{TAut}_0(K\langle x, y, z\rangle)$, i.e., the group of Ind-scheme automorphisms, and prove Theorem 3.1.13. We use the approximation results of Sec. 3.3.3. In what follows, we assume that $\mathrm{char}(K) \neq 2$. As in the preceding chapter, $\{x, y, z\}_*$ denotes the associator of x, y, and z with respect to a fixed binary linear operation $*$, i.e.,

$$\{x, y, z\}_* = (x * y) * z - x * (y * z).$$

Proposition 3.5.11. *Let $\Psi \in \mathrm{Aut}_{\mathrm{Ind}}(\mathrm{TAut}_0(K\langle x, y, z\rangle))$ stabilize all linear automorphisms. Let*

$$\phi : x \to x,\ y \to y,\ z \to z + xy.$$

Then either

$$\Psi(\phi) : x \to x,\ y \to y,\ z \to z + axy \quad \text{or} \quad \Psi(\phi) : x \to x,\ y \to y,\ z \to z + byx$$

for some $a, b \in K$.

Proof. Consider the automorphism

$$\phi : x \to x,\ y \to y,\ z \to z + xy.$$

Then

$$\Psi(\phi) : x \to x,\ y \to y,\ z \to z + x * y,$$

where $x * y = axy + byx$. Let $a \neq 0$. We can make the star product $* = *_{a,b}$ into $x * y = xy + \lambda yx$ by conjugation with the mirror anti-automorphism and appropriate linear substitution. Therefore, we need to prove that $\lambda = 0$, which implies $\Psi(\phi) = \phi$. □

The following two lemmas are proved by straightforward computation.

Lemma 3.5.12. *Let $A = K\langle x, y, z\rangle$. Let $f * g = fg + \lambda fg$. Then*

$$\{f, g, h\}_* = \lambda[g, [f, h]].$$

In particular,

$$\{f, g, f\}_* = 0, \quad f * (f * g) - (f * f) * g = -\{f, f, g\}_* = \lambda[f, [f, g]],$$
$$(g * f) * f - g * (f * f) = \{g, f, f\}_* = \lambda[f, [f, g]].$$

Lemma 3.5.13. *Let*

$$\varphi_1 : x \to x + yz,\ y \to y,\ z \to z; \quad \varphi_2 : x \to x,\ y \to y,\ z \to z + yx; \quad \varphi = \varphi_2^{-1}\varphi_1^{-1}\varphi_2\varphi_1.$$

Then modulo terms of order $\geqslant 4$ *we have*

$$\varphi : x \to x + y^2x,\ y \to y,\ z \to z - y^2z \quad \text{and} \quad \Psi(\varphi) : x \to x + y * (y * x),\ y \to y,\ z \to z - y * (y * z).$$

Lemma 3.5.14.

(a) *Let $\phi_l : x \to x$, $y \to y$, $z \to z + y^2x$. Then*

$$\Psi(\phi_l) : x \to x,\ y \to y,\ z \to z + y * (y * x).$$

(b) *Let $\phi_r : x \to x,\ y \to y,\ z \to z + xy^2$. Then*

$$\Psi(\phi_r) : x \to x,\ y \to y,\ z \to z + (x * y) * y.$$

Proof. According to the results of the previous section we have

$$\Psi(\phi_l) : x \to x,\ y \to y,\ z \to z + P(y, x),$$

where $P(y, x)$ is homogenous of degree 2 with respect to y and degree 1 with respect to x. We have to prove that $H(y, x) = P(y, x) - y * (y * x) = 0$.

Let

$$\tau : x \to z,\ y \to y,\ z \to x; \quad \tau = \tau^{-1}, \quad \phi' = \tau\phi_l\tau^{-1} : x \to x + y^2z,\ y \to y,\ z \to z.$$

Then

$$\Psi(\phi_l') : x \to x + P(y, z),\ y \to y,\ z \to z.$$

Let

$$\phi_l'' = \phi_l\phi_l' : x \to x + P(y, z),\ y \to y,\ z \to z + P(y, x)$$

modulo terms of degree $\geqslant 4$.

Let $\tau : x \to x - z$, $y \to y$,$z \to z$ and φ_2 and φ be the automorphisms described in Lemma 3.5.13. Then

$$T = \tau^{-1}\phi_l^{-1}\tau\phi_l'' : x \to x,\ y \to y,\ z \to z$$

modulo terms of order $\geqslant 4$. On the other hand,

$$\Psi(T) : x \to x + H(y, z) - H(y, x),\ y \to y,\ z \to z + P$$

modulo terms of order $\geqslant 4$. Since $\deg_y(H(y, x) = 2$, $\deg_x(H(y, x)) = 1$, we get $H = 0$.

The proof of (b) is similar. □

Lemma 3.5.15.

(a) *Let*

$$\psi_1 : x \to x + y^2,\ y \to y,\ z \to z; \quad \psi_2 : x \to x,\ y \to y,\ z \to z + x^2.$$

Then

$$[\psi_1, \psi_2] = \psi_2^{-1}\psi_1^{-1}\psi_2\psi_1 : x \to x,\ y \to y,\ z \to z + y^2x + xy^2,$$
$$\Psi([\psi_1, \psi_2]) : x \to x,\ y \to y,\ z \to z + (y * y) * x + x * (y * y).$$

(b) *We have*

$$\phi_l^{-1}\phi_r^{-1}[\psi_1, \psi_2] : x \to x,\ y \to y,\ z \to z$$

modulo terms of order $\geqslant 4$ *but*

$$\Psi\left(\phi_l^{-1}\phi_r^{-1}[\psi_1, \psi_2]\right) : x \to x,\ y \to y,$$
$$z \to z + (y * y) * x + x * (y * y) - (x * y) * y - y * (y * x) = z + 4\lambda[x[x, y]]$$

modulo terms of order $\geqslant 4$.

Proof. The assertion (a) can be obtained by direct computation; the assertion (b) follows from (a) and Lemma 3.5.12. □

Proposition 3.5.11 follows from Lemma 3.5.15.

We need several auxiliary lemmas. The first lemma is an analog of the hiking procedure from [34, 122].

Lemma 3.5.16. *Let K be algebraically closed and let $n_1, \dots, n_m$ be positive integers. Then there exist $k_1, \dots, k_s \in \mathbb{Z}$ and $\lambda_1, \dots, \lambda_s \in K$ such that*

(i) $\sum k_i = 1$ *modulo* $\mathrm{char}(K)$ (*if* $\mathrm{char}(K) = 0$, *then* $\sum k_i = 1$).

(ii) $\sum_i k_i^{n_j}\lambda_i = 0$ *for all* $j = 1, \dots, m$.

For $\lambda \in K$, we define an automorphism $\psi_\lambda : x \to x$, $y \to y$, $z \to \lambda z$.

The next lemma provides for some translation between the language of polynomials and the group action language. It is similar to the hiking process (see [34, 122]).

Lemma 3.5.17. *Let $\varphi \in K\langle x, y, z\rangle$. Let $\varphi(x) = x$, $\varphi(y) = y + \sum_i R_i + R'$, and $\varphi(z) = z + Q$. Let $\deg(R_i) = N$, let the degrees of all monomials in R' be greater than N, and let the degrees of all monomials in Q be greater than or equal to N. Finally, assume that $\deg_z(R_i) = i$ and the z-degree of all monomials of R_1 are greater than 0. Then the following assertions hold*:

(a) $\psi_\lambda^{-1}\varphi\psi_\lambda : x \to x$, $y \to y + \sum_i \lambda^i R_i + R''$, $z \to z + Q'$. *Also the total degree of all monomials involving R' is greater than N, and the degree of all monomials of Q is greater than or equal to N.*

(b) *Let* $\phi = \prod \left(\psi_{\lambda_i^{-1}}\varphi\psi_{\lambda_i}\right)^{k_i}$. *Then*

$$\phi : x \to x,\ y \to y + \sum_i R_i\lambda_i^{k_i} + S,\ z \to z + T,$$

where the degree of all monomials of S is greater than N and the degree of all monomials of T is greater than or equal to N.

Proof. Assertion (a) is proved by adirect computation and (b) is a consequence of (a). □

Remark 3.5.18. In the case of zero characteristic, the condition of K being algebraically closed can be dropped. After hiking for several steps, we must prove the following assertion.

Lemma 3.5.19. *Let $\mathrm{char}(K) = 0$ and n be a positive integer. Then there exist $k_1, \dots, k_s \in \mathbb{Z}$ and $\lambda_1, \dots, \lambda_s \in K$ such that*

(i) $\sum k_i = 1$;
(ii) $\sum_i k_i^n \lambda_i = 0$.

Using this lemma, we can cancel all terms in the product in Lemma 3.5.17 except for constant terms. The proof of Lemma 3.5.19 for any field of zero characteristic can be obtained by using the following observation.

Lemma 3.5.20.

$$\left(\sum_{i=1}^{n} \lambda_i\right)^n - \sum_j \left(\lambda_1 + \cdots + \widehat{\lambda_j} + \cdots + \lambda_n\right)^n + \cdots + \\ + (-1)^{n-k} \sum_{i_1<\cdots<i_k} (x_{i_1} + \cdots + x_{i_k})^n + \cdots + (-1)^{n-1} (x_1^n + \cdots + x_n^n) = n! \prod_{i=1}^{n} x_i,$$

and if $m < n$, then

$$\left(\sum_{i=1}^{n} \lambda_i\right)^m - \sum_j \left(\lambda_1 + \cdots + \widehat{\lambda_j} + \cdots + \lambda_n\right)^m + \cdots + \\ + (-1)^{n-k} \sum_{i_1<\cdots<i_k} (x_{i_1} + \cdots + x_{i_k})^m + \cdots + (-1)^{n-1} (x_1^m + \cdots + x_n^m) = 0.$$

Lemma 3.5.20 allows one to replace the nth powers by product of constants; after this Lemma 3.5.19 becomes obvious.

Lemma 3.5.21. *Let $\varphi : x \to x+R_1$, $y \to y+R_2$, $z \to z'$, such that the total degree of all monomials in R_1 and R_2 is greater than or equal to N. Then for $\Psi(\varphi) : x \to x + R_1'$, $y \to y + R_2'$, $z \to z''$ with the total degree of all monomials in R_1' and R_2' also greater than or equal to N.*

The proof is similar to the proof of Theorem 3.3.6.

Lemmas 3.5.21, 3.5.17, and 3.5.16 imply the following statement.

Lemma 3.5.22. *Let $\varphi_j \in \mathrm{Aut}_0(K\langle x, y, z\rangle)$, $j = 1, 2$, such that*

$$\varphi_j(x) = x, \quad \varphi_j(y) = y + \sum_i R_i^j + R_j', \quad \varphi_j(z) = z + Q_j.$$

Let $\deg(R_i^j) = N$. Assume that the degree of all monomials in R_j' is greater than N, while the degree of all monomials in Q is greater than or equal to N; $\deg_z(R_i) = i$, and the z-degree of all monomials in R_1 is positive. Let $R_0^1 = 0$ and $R_0^2 \neq 0$. Then $\Psi(\varphi_1) \neq \varphi_2$.

Consider the automorphism

$$\phi : x \to x,\ y \to y,\ z \to z + P(x, y).$$

Let $\Psi \in \mathrm{Aut}_{\mathrm{Ind}}\, \mathrm{TAut}_0(k\langle x, y, z\rangle)$ stabilize the standard torus action pointwise. Then

$$\Psi(\phi) : x \to x,\ y \to y,\ z \to z + Q(x, y).$$

We introduce the notation $\bar{\Psi}(P) = Q$. We prove that $\bar{\Psi}(P) = P$ for all P if Ψ stabilizes all linear automorphisms and $\bar{\Psi}(xy) = xy$. We proceed by strong induction on the total degree. The base of induction corresponds to $k = 1$ and $l = 1$.

Lemma 3.5.23. *If $\bar{\Psi}(P) = P$ for all monomials $P(x, y)$ of total degree $<k+l$, then $\bar{\Psi}(x^k y^l) = x^k y^l$.*

Proof. Let

$$\phi : x \to x,\ y \to y,\ z \to z + x^k y^l, \quad \varphi_1 : x \to x + y^l,\ y \to y,\ z \to z,$$
$$\varphi_2 : x \to x,\ y \to y + x^k,\ z \to z, \quad \varphi_3 : x \to x,\ y \to y,\ z \to z + xy,$$
$$h : x \to x,\ y \to y,\ z \to z - x^{k+1}.$$

Then for $k > 1$ and $l > 1$ we have

$$g = h\varphi_3^{-1}\varphi_1^{-1}\varphi_2^{-1}\varphi_3\varphi_1\varphi_2 :$$
$$x \mapsto x - y^l + (y - (x - y^l)^k)^l, \quad y \mapsto y - (x - y^l)^k + (x - y^l + (y - (x - y^l)^k)^l)^k,$$
$$z \mapsto z - xy - x^{k+1} + (x - y^l)(y - (x - y^l)^k).$$

Observe that the heights of $g(x) - x$, $g(y) - y$, and $g(z) - z$ are at least $k + l - 1$ for $k > 1$ or $l > 1$. Then we use Theorem 3.3.6 and the induction step. Applying Ψ implies the result since $\Psi(\varphi_i) = \varphi_i$, $i = 1, 2, 3$, and $\varphi(H_N) \subseteq H_N$ for all N. □

Let

$$M_{k_1,\dots,k_s} = x^{k_1}y^{k_2}\cdots y^{k_s} \quad \text{or} \quad M_{k_1,\dots,k_s} = x^{k_1}y^{k_2}\cdots x^{k_s}$$

for even and odd s, respectively, and $k = \sum_{i=1}^{n} k_i$. Then

$$M_{k_1,\dots,k_s} = M_{k_1,\dots,k_{s-1}}y^{k_s} \quad \text{or} \quad M_{k_1,\dots,k_s} = M_{k_1,\dots,k_{s-1}}x^{k_s}$$

for even s and odd s, reapectively.

We prove that $\bar\Psi(M_{k_1,\dots,k_s}) = M_{k_1,\dots,k_s}$. By induction we assume that $\bar\Psi(M_{k_1,\dots,k_{s-1}}) = M_{k_1,\dots,k_{s-1}}$.

For any monomial $M = M(x, y)$, we define the automorphism

$$\varphi_M : x \to x,\ y \to y,\ z \to z + M$$

and the automorphisms

$$\phi_k^e : x \to x,\ y \to y + zx^k,\ z \to z \quad \text{and} \quad \phi_k^o : x \to x + zy^k,\ y \to y,\ z \to z.$$

We present the case of even s (the case for odd s is similar).

Let $D^e_{zx^k}$ be a derivation of $K\langle x, y, z\rangle$ such that

$$D^e_{zx^k}(x) = 0, \quad D^e_{zx^k}(y) = zx^k, \quad D^e_{zx^k}(z) = 0.$$

Similarly, let $D^o_{zy^k}$ be a derivation of $K\langle x, y, z\rangle$ such that

$$D^o_{zy^k}(y) = 0, \quad D^o_{zx^k}(x) = zy^k, \quad D_{zy^k}(z)^o = 0.$$

The following lemma is proved by a direct computation.

Lemma 3.5.24. *Let*

$$u = \phi^{e\ -1}_{k_s}\varphi(M_{k_1,\dots,k_{s-1}})^{-1}\phi^e_{k_s}\varphi(M_{k_1,\dots,k_{s-1}})$$

for even s and

$$u = \phi^{o\ -1}_{k_s}\varphi(M_{k_1,\dots,k_{s-1}})^{-1}\phi^o_{k_s}\varphi(M_{k_1,\dots,k_{s-1}})$$

for odd s. Then

$$u : x \to x,\ y \to y + M_{k_1,\dots,k_s} + N',\ z \to z + D^e_{zx^k}(M_{k_1,\dots,k_{s-1}}) + N$$

for even s and

$$u : x \to x + M_{k_1,\dots,k_s} + N',\ y \to y,\ z \to z + D^o_{zx^k}(M_{k_1,\dots,k_{s-1}}) + N$$

for odd s, where N and N' are sums of terms of degree $>k = \sum_{i=1}^{s} k_i$.

Let

$$\psi(M_{k_1,\dots,k_s}) : x \to x,\ y \to y,\ z \to z + M_{k_1,\dots,k_s},$$
$$\alpha_e : x \to x,\ y \to y - z,\ z \to z,$$
$$\alpha_o : x \to x - z,\ y \to y,\ z \to z.$$

Let $P_M = \Psi(M) - M$. We prove that $P_M = 0$. Let

$$v = \psi(M_{k_1,\dots,k_s})^{-1}\alpha_e\psi(M_{k_1,\dots,k_s})u\alpha_e^{-1} \quad \text{or} \quad v = \psi(M_{k_1,\dots,k_s})^{-1}\alpha_o\psi(M_{k_1,\dots,k_s})u\alpha_o^{-1}$$

for even and odd s, respectively.

The following lemma is also proved by a direct computation.

Lemma 3.5.25.

(a) *We have*

$$v : x \to x,\ y \to y + H,\ z \to z + H_1 + H_2$$

for even s *and*

$$v : x \to x + H,\ y \to y,\ z \to z + H_1 + H_2$$

for odd s.

(b) *We have*

$$\Psi(v) : x \to x,\ y \to y + P_{M_{k_1,\dots,k_s}} + \widetilde{H},\ z \to z + \widetilde{H_1} + \widetilde{H_2}$$

for even s *and*

$$\Psi(v) : x \to x + P_{M_{k_1,\dots,k_s}} + \widetilde{H},\ y \to y,\ z \to z + \widetilde{H_1} + \widetilde{H_2}$$

for odd s, *where* H_2 *and* $\widetilde{H_2}$ *are the sums of terms of degree greater than* $k = \sum\limits_{i=1}^{s} k_i$, H *and* $\widetilde{H}$ *are the sums of terms of degree* $\geqslant k$ *and positive* z-*degree, and* H_1 *and* $\widetilde{H_1}$ *are the sums of terms of degree* k *and positive* z-*degree.*

Proof of Theorem 3.1.13. The assertion (b) follows from (a). To prove (a), we show that $\bar{\Psi}(M) = M$ for any monomial $M(x, y)$ and for any $\Psi \in \mathrm{Aut}_{\mathrm{Ind}}(\mathrm{TAut}(\langle x, y, z\rangle))$ stabilizing the standard torus action T^3 and ϕ. The automorphism $\Psi(\Phi_M)$ has the form described in Lemma 3.5.25. But in this case Lemma 3.5.22 implies $\bar{\Psi}(M) - M = 0$. □

3.6. Some Open Questions Concerning the Tame Automorphism Group

As the conclusion of the paper, we would like to raise the following questions.

1. Is it true that any automorphism φ of $\mathrm{Aut}(K\langle x_1, \dots, x_n\rangle)$ (in the group-theoretic sense, that is, not necessarily an automorphism preserving the Ind-scheme structure) for $n = 3$ is semi-inner, i.e., is a conjugation by some automorphism or mirror anti-automorphism?
2. Is it true that $\mathrm{Aut}(K\langle x_1, \dots, x_n\rangle)$ is generated by affine automorphisms and the automorphism $x_n \to x_n + x_1x_2$, $x_i \to x_i$, $i \neq n$? For $n \geqslant 5$, it seems to be easier and the answer is probably positive; however, for $n = 3$ the answer is known to be negative (cf. [82,193]). For $n \geqslant 4$, we believe the answer is positive.
3. Is it true that $\mathrm{Aut}(K[x_1, \dots, x_n])$ is generated by linear automorphisms and automorphism $x_n \to x_n + x_1x_2$, $x_i \to x_i$, $i \neq n$? For $n = 3$, the answer is negative (see the proof of the Nagata conjecture; [180, 183, 200]). For $n \geqslant 4$, it is plausible that the answer is positive.
4. Is any automorphism φ of $\mathrm{Aut}(K\langle x, y, z\rangle)$ (in the group-theoretic sense) semi-inner?
5. Is it true that the conjugation in Theorems 3.1.8 and 3.1.12 can be done by some tame automorphism? Assume that $\psi^{-1}\varphi\psi$ is tame for any tame φ. Does it follow that ψ is tame?
6. Prove Theorem 3.1.13 for $\mathrm{char}(K) = 2$. Does it hold on the set-theoretic level, i.e., $\mathrm{Aut}(\mathrm{TAut}(K\langle x, y, z\rangle))$ are generated by conjugations by an automorphism or the mirror anti-automorphism?

Similar questions can be formulated for nice automorphisms.

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