UDK 517.98
DOI: 10.20310/1810-0198-2017-22-6-1218-1228

CANONICAL AND BOUNDARY REPRESENTATIONS ON THE LOBACHEVSKY PLANE ASSOCIATED WITH LINEAR BUNDLES

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We describe canonical representations on the Lobachevsky plane, associated with sections of linear bundles, corresponding boundary representations and Poisson and Fourier transforms.
Keywords: Lobachevsky plane; canonical representations; distributions; boundary representations; Poisson and Fourier transforms

We give a generalization of the work [1] where we studied canonical and boundary representations of the group $G = \mathrm{SU}(1,1)$ on the Lobachevsky plane D. Canonical representations in [1] are deformations of the quasi-regular representation U of G on D. Now we study similar deformations of representations of G in the space sections of linear bundles on D, or, what is the same, deformations of representations of G induced by characters of a maximal compact subgroup K. See also our note [2].

§ 1. Canonical representations associated with a character of K

The Lobachevsky plane is the unit disk $D\colon z\overline{z}<1$ on the complex plane with the linear-fractional action of G:

$$z \mapsto z\cdot g = \frac{az+\overline{b}}{bz+\overline{a}}, \quad g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad a\overline{a}-b\overline{b}=1.$$

The boundary S of D is the circle $z\overline{z}=1$, it consists of points $s=\exp i\alpha$, the measure ds on S is $d\alpha$. Let $\overline{D}$ be the closure of D: $\overline{D}=D\cup S$. The stabilizer of the point $z=0$ is the maximal compact subgroup $K=\mathrm{U}(1)$ consisting of diagonal matrices:

$$k = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}, \quad a\overline{a}=1, \tag{1.1}$$

so that $D=G/K$. Recall principal non-unitary series representations of G trivial on the center. Let $\sigma\in\mathbb{C}$. The representation T_σ acts on the space $\mathcal{D}(S)$ by

$$(T_\sigma(g)\varphi)(s) = \varphi(s\cdot g)|bs+\overline{a}|^{2\sigma}.$$

If $\sigma \notin \mathbb{Z}$, then T_σ is irreducible and equivalent to $T_{-\sigma-1}$ (for $\sigma \in \mathbb{Z}$ there is a "partial equivalence"). The following operator A_σ acts on $\mathcal{D}(S)$ and intertwines T_σ and $T_{-\sigma-1}$:

$$\left(A_\sigma \varphi\right)(s) = \int_S |1 - s\overline{u}|^{-2\sigma-2} \varphi(u) du, \tag{1.2}$$

exponents s^n are eigenfunctions for A_σ with eigenvalues $a_n(\sigma)$:

$$a_n(\sigma) = 2\pi \, (-1)^n \, \frac{\Gamma(-2\sigma - 1)}{\Gamma(-\sigma + n)\, \Gamma(-\sigma - n)} \, . \tag{1.3}$$

We shall use denotation

$$a^{[q]} = a(a+1)\ldots(a+q-1), \quad z^{\mu,\,n} = |z|^\mu \left(\frac{z}{|z|}\right)^n, \;\; \mu \in \mathbb{C}, \;\; n \in \mathbb{Z}, \;\; q \in \mathbb{N}.$$

Let $\lambda \in \mathbb{C}$, $m \in \mathbb{Z}$. We define the *canonical representation* $R_{\lambda,m}$ of the group G as follows:

$$\left(R_{\lambda,m}(g) f\right)(z) = f(z \cdot g)\, (bz + \overline{a})^{-2\lambda-4, 2m},$$

it acts on the space $\mathcal{D}(\overline{D})$. This space consists of functions $f(z) = f(z, \overline{z})$ such that for any $f \in \mathcal{D}(\overline{D})$ there is a neighbourhood U of $\overline{D}$ such that f belongs to $\mathcal{D}(U)$. The representation $R_{\lambda,m}$ is the restriction to G of a representation of the overgroup $\widetilde{G} = \mathrm{SL}(2, \mathbb{C})$.

Introduce the inner product

$$\langle f, \, h \rangle_D = \int_D f(z)\, \overline{h(z)}\, dxdy, \;\; z = x + iy. \tag{1.4}$$

It is invariant with respect to the pair $(R_{\lambda,m}, \, R_{-\overline{\lambda}-2,m})$:

$$\langle R_{\lambda,m}(g) f, \, h \rangle_D = \langle f, \, R_{-\overline{\lambda}-2,m}(g^{-1}) h \rangle_D, \tag{1.5}$$

where $g \in G$. Let us define the operator $Q_{\lambda,m}$ – first on $\mathcal{D}(D)$:

$$\left(Q_{\lambda,m} f\right)(z) = c(\lambda, m) \int_D (1 - z\overline{w})^{2\lambda, 2m} f(w) dudv,$$

where

$$c(\lambda, m) = \frac{-\lambda + m - 1}{\pi} \, .$$

It intertwines $R_{\lambda,m}$ and $R_{-\lambda-2,m}$:

$$Q_{\lambda,m}\, R_{\lambda,m}(g) = R_{-\lambda-2,m}(g)\, Q_{\lambda,m}, \quad g \in G,$$

and interacts with the form (1.4) as follows:

$$\langle Q_{\lambda,m} f, \, h \rangle_D = \langle f, \, Q_{\overline{\lambda},m}\, h \rangle_D. \tag{1.6}$$

The formulae (1.5) and (1.6) allow to extend the representation $R_{\lambda,m}$ and the operator $Q_{\lambda,m}$ to the space $\mathcal{D}'(\overline{D})$ of distributions on $\overline{D}$.

§ 2. Boundary representations

Any canonical representation of the group G generates 2 representations related to the boundary S of the disk D (see $L_{\lambda,m}$ and $M_{\lambda,m}$ below).

Introduce on $\mathbb{C}$ polar coordinates (r,s): $z=rs$, $r\geqslant 0$, $s\in S$. Let

$$p = 1 - z\overline{z} = 1 - r^2,$$

so that $D=\{p>0\}$ and $S=\{p=0\}$. The Euclidean measure $dxdy$ on D is $(1/2)\,dp\,ds$. Consider the Taylor series of $f\in\mathcal{D}(\overline{D})$ in powers of p:

$$f(z) \sim a_0 + a_1\,p + a_2\,p^2 + \cdots, \tag{2.1}$$

where $a_k = a_k(s)$ are functions in $\mathcal{D}(S)$:

$$a_k(s) = \frac{1}{k!}\left(\frac{\partial}{\partial p}\right)^k\Big|_{p=0} f(z).$$

Denote by $\Sigma_k(\overline{D})$ the space of distributions on $\mathbb{C}$ concentrated at S and of the form

$$\zeta = \varphi_0(s)\,\delta(p) + \varphi_1(s)\,\delta'(p) + \cdots + \varphi_k(s)\,\delta^{(k)}(p), \tag{2.2}$$

where $\delta(p)$ is the Dirac delta function on the real line (being a continuous linear functional on $\mathcal{D}(\mathbb{R})$) and $\delta^{(j)}(p)$ its j-th derivative. Set

$$\Sigma(\overline{D}) = \cup_{k=0}^{\infty}\,\Sigma_k(\overline{D}).$$

There is a natural filtration

$$\Sigma_0(\overline{D}) \subset \Sigma_1(\overline{D}) \subset \Sigma_2(\overline{D}) \subset \cdots \tag{2.3}$$

A distribution $\varphi(s)\,\delta^{(l)}(p)$ acts on a function $f\in\mathcal{D}(\overline{D})$ as follows:

$$\langle\varphi(s)\,\delta^{(l)}(p),\, f\rangle = \frac{1}{2}(-1)^l\, l!\,\langle\varphi,\, a_l\rangle_S.$$

The canonical representation $R_{\lambda,m}$ acting on $\mathcal{D}'(\overline{D})$, preserves the space $\Sigma\,(\overline{D})$ and the filtration (2.3). Denote by $L_{\lambda,m}$ the restriction of $R_{\lambda,m}$ to $\Sigma\,(\overline{D})$. Let us assign to the distribution (2.2) the column $(\varphi_0,\varphi_1,\ldots,\varphi_k,0,0,\ldots)$.

Lemma 2.1. *On these columns the representation $L_{\lambda,m}$ is a upper triangular matrix. It is equivalent to a upper triangular matrix with diagonal $T_{-\lambda-1}$, $T_{-\lambda}$, $T_{-\lambda+1}$, ... The equivalence is given by multiplication of the functions $\varphi_k(s)$ by s^{-m}.*

Proof. Set $\varphi_k(s)\,s^{-m}=\psi_k(s)$. We have to trace how the operator $L_{\lambda,m}(g)$ $(g\in G)$ acts on the distribution $\psi_k(s)s^m\cdot\delta^{(k)}(p)$. This distribution is mapped on

$$\psi_k(\widetilde{s})\,\widetilde{s}^m\,\delta^{(k)}(\widetilde{p})\,(bz+\overline{a})^{-2\lambda-4,2m}. \tag{2.4}$$

Since $\widetilde{p}=p\cdot|bz+\overline{a}|^{-2}$ and $\delta^{(k)}(p)$ is homogeneous of degree $-k-1$, the distribution (2.4) is equal to

$$\begin{aligned}&\psi_k(\widetilde{s})\,\widetilde{s}^m\,(bz+\overline{a})^{-2\lambda-2-2k,2m}\,\delta^{(k)}(p)\\ &= \psi_k(\widetilde{s})\,\widetilde{s}^m\,(bs+\overline{a})^{-2\lambda-2-2k,2m}\,\delta^{(k)}(p) + \cdots,\end{aligned} \tag{2.5}$$

where the dots means a distribution in $\Sigma_{k-1}(\overline{D})$. Since $\overline{s}=s^{-1}$, we have

$$\widetilde{s} = \frac{as+\overline{b}}{bs+\overline{a}} = s\cdot\frac{\overline{b}\overline{s}+a}{bs+\overline{a}} = s\cdot(bs+\overline{a})^{0,-2}, \tag{2.6}$$

so that the distribution (2.5) is equal to

$$\psi_k(\widetilde{s})\,|bs+\overline{a}|^{-2\lambda-2+2k}\cdot s^m\,\delta^{(k)}(p)+\cdots.$$

The factor in front of $s^m\,\delta^{(k)}(p)$ is precisely $(T_{-\lambda-1+k}(g)\psi_k)\,(s)$. □

For $f\in\mathcal{D}(\overline{D})$, let $a(f)$ denote the column $(a_0,a_1,\ldots)$ of the Taylor coefficients of f, see (2.1). The representation $M_{\lambda,m}$ acts on these columns by:

$$M_{\lambda,m}(g)\,a(f)=a(R_{\lambda,m}(g)f).$$

Lemma 2.2. *The representation $M_{\lambda,m}$ is a lower triangular matrix. It is equivalent to a lower triangular matrix with diagonal $T_{-\lambda-2}, T_{-\lambda-3},\ldots$. The equivalence is given by multiplication of the Taylor coefficients $a_k(s)$ by s^{-m}.*

Proof. By expanding functions in a Taylor series, we find that the k-th Taylor coefficient of the function $f^g(z)=(R_{\lambda,m}(g)f)(z)$ is

$$\begin{aligned} & a_k(\widetilde{s})\,(bs+\overline{a})^{-2\lambda-4-2k,2m}+\cdots \\ &= a_k(\widetilde{s})\,\widetilde{s}^{-m}\,|bs+\overline{a}|^{-2\lambda-4-2k}\cdot s^m+\cdots, \end{aligned} \tag{2.7}$$

where the dots means a linear combination of $a_0(\widetilde{s}),\ldots,a_{k-1}(\widetilde{s})$ whose coefficients are some functions of s. Here we used again that $\widetilde{p}=p\cdot|bz+\overline{a}|^{-2}$ and formula (2.6). Now setting $a_k(s)=d_k(s)\,s^m$, we see from (2.7) that the coefficient $d_k^g(s)$ for $f^g(s)$ is $(T_{-\lambda-2-k}(g)\,d_k)\,(s)+\cdots$. □

§ 3. Poisson transform

Let $\lambda,\,\sigma\in\mathbb{C}$ and $m\in\mathbb{Z}$. We define the *Poisson transform associated with the canonical representation* $R_{\lambda,m}$ as the map $P_{\lambda,\sigma}^{(m)}:\mathcal{D}(S)\to C^\infty(D)$ by the following formula

$$\left(P_{\lambda,\sigma}^{(m)}\,\varphi\right)(z)=p^{-\lambda-\sigma-2}\int_S(1-s\overline{z})^{2\sigma,-2m}\,s^m\,\varphi(s)\,ds. \tag{3.1}$$

Theorem 3.1. *The Poisson transform $P_{\lambda,\sigma}^{(m)}$ intertwines the representations $T_{-\sigma-1}$ and the canonical representation $R_{\lambda,m}$:*

$$R_{\lambda,m}(g)\,P_{\lambda,\sigma}^{(m)}=P_{\lambda,\sigma}^{(m)}\,T_{-\sigma-1}(g)\quad(g\in G).$$

Theorem 3.2. *With the intertwining operators A_σ and $Q_{\lambda,m}$ the Poisson transform $P_{\lambda,\sigma}^{(m)}$ interacts as follows:*

$$P_{\lambda,\sigma}^{(m)}\,A_\sigma=a_{-m}(\sigma)\,P_{\lambda,-\sigma-1}^{(m)}, \tag{3.2}$$

$$Q_{\lambda,m}\,P_{\lambda,\sigma}^{(m)}=\Lambda^{(m)}(\lambda,\sigma)\,P_{-\lambda-2,\sigma}^{(m)}, \tag{3.3}$$

where

$$\Lambda^{(m)}(\lambda,\sigma)=\frac{\Gamma(-\lambda+\sigma)\,\Gamma(-\lambda-\sigma-1)}{\Gamma(-\lambda-m)\,\Gamma(-\lambda+m-1)}.$$

Proof. Formula (3.2) follows immediately from (3.1). Let us prove (3.3). Applying the operator $Q_{\lambda,m}\,P_{\lambda,\sigma}^{(m)}$ to a function $\varphi\in\mathcal{D}(S)$, we get the multiple integral:

$$\begin{aligned} \left(Q_{\lambda,m}\,P_{\lambda,\sigma}^{(m)}\,\varphi\right)(z) &= c(\lambda,m)\int_D(1-z\overline{w})^{2\lambda,2m}\,(1-w\overline{w})^{-\lambda-\sigma-2}dudv \\ &\times\int_S(1-s\overline{w})^{2\sigma,-2m}\,s^m\,\varphi(s)\,ds,\quad w=u+iv \end{aligned} \tag{3.4}$$

By (3.1), the function $\left(P^{(m)}_{\lambda,\sigma}\,\varphi\right)(z)$ behaves as $C_1\,p^{-\lambda-\sigma-2}+C_2\,p^{-\lambda+\sigma-1}$ when $p\to 0$. Therefore, the integral (3.4) converges absolutely for $\mathrm{Re}\,\sigma>-1/2$, $\mathrm{Re}\,(\lambda+\sigma)<-1$, $\mathrm{Re}\,(-\lambda+\sigma)>0$, and we can then change the order of integration. We obtain

$$\left(Q_{\lambda,m}\,P^{(m)}_{\lambda,\sigma}\,\varphi\right)(z)=c(\lambda,m)\int_S K(z,s)\,s^m\,\varphi(s)\,ds, \tag{3.5}$$

where the kernel $K(z,s)$ is given by

$$K(z,s)=\int_D (1-z\overline{w})^{2\lambda,2m}\,(1-w\overline{w})^{-\lambda-\sigma-2}\,(1-s\overline{w})^{2\sigma,-2m}\,dudv.$$

Let us compute it. Using the formula

$$1-\widetilde{z}\overline{\widetilde{w}}=\frac{1-z\overline{w}}{(bz+\overline{a})(\overline{b}\overline{w}+a)}$$

and similar formulae with replacing z by s and by w, we find that the kernel $K(z,s)$ has the following invariance property:

$$K(\widetilde{z},\widetilde{s})\,(bz+\overline{a})^{2\lambda,2m}\,(bs+\overline{a})^{2\sigma,-2m}=K(z,s).$$

Take here $z=0,\ s=1$ and write z and s instead of $\widetilde{z}$ and $\widetilde{s}$ respectively. Then we have

$$K(z,s)=K(0,1)\,a^{-2\lambda,2m}\,(b+\overline{a})^{-2\sigma,2m} \tag{3.6}$$

and

$$z=\frac{\overline{b}}{\overline{a}},\quad s=\frac{a+\overline{b}}{b+\overline{a}}.$$

For these z and s we find

$$1-z\overline{z}=\frac{1}{a\overline{a}},\quad 1-s\overline{z}=1-\frac{a+\overline{b}}{b+\overline{a}}\cdot\frac{b}{a}=\frac{1}{a(b+\overline{a})},$$

so that

$$a^{-2\lambda,2m}\,(b+\overline{a})^{-2\sigma,2m}=(1-z\overline{z})^{\lambda-\sigma}\,(1-s\overline{z})^{2\sigma,-2m}. \tag{3.7}$$

It remains to compute $K(0,1)$:

$$K(0,1)=\int_D (1-w\overline{w})^{-\lambda-\sigma-2}\,(1-w)^{2\sigma,2m}\,dudv. \tag{3.8}$$

Expand $(1-w)^{2\sigma,2m}$ in a binomial series:

$$\begin{aligned}(1-w)^{2\sigma,2m} &= (1-w)^{\sigma+m}\,(1-\overline{w})^{\sigma-m}\\ &= \sum_{q,j=0}^{\infty}\binom{\sigma+m}{q}\binom{\sigma-m}{j}(-1)^{q+j}\,w^q\,\overline{w}^j.\end{aligned}$$

A non-zero contribution to (3.8) is given by the terms with $q=j$ only, so that

$$K(0,1)=\sum_{q=0}^{\infty}\binom{\sigma+m}{q}\binom{\sigma-m}{q}\int_D (1-w\overline{w})^{-\lambda-\sigma-2}\,(w\overline{w})^q\,dudv.$$

The latter integral is equal to $B(q+1,-\lambda-\sigma-1)$, so that

$$\begin{aligned}
K(0,1) &= \pi \sum_{q=0}^{\infty} \frac{(\sigma+m)^{[q]}\,(\sigma-m)^{[q]}}{q!} \cdot \frac{\Gamma(-\lambda-\sigma-1)}{\Gamma(-\lambda-\sigma+q)} \\
&= \pi \sum_{q=0}^{\infty} \frac{(-\sigma-m)^{[q]}\,(-\sigma+m)^{[q]}}{(-\lambda-\sigma-1)^{[q+1]}\,q!} \\
&= \frac{\pi}{-\lambda-\sigma-1} \sum_{q=0}^{\infty} \frac{(-\sigma-m)^{[q]}\,(-\sigma+m)^{[q]}}{(-\lambda-\sigma)^{[q]}\,q!} \\
&= \frac{\pi}{-\lambda-\sigma-1}\, F(-\sigma-m,-\sigma+m;-\lambda-\sigma;1) \\
&= \pi \, \frac{\Gamma(-\lambda-\sigma-1)\,\Gamma(-\lambda+\sigma)}{\Gamma(-\lambda+m)\,\Gamma(-\lambda-m)} \\
&= \frac{1}{c(\lambda,m)}\,\Lambda^{(m)}(\lambda,\sigma).
\end{aligned}$$

Thus, collecting (3.5), (3.6) and (3.7), we obtain

$$\left(Q_{\lambda,m}\, P_{\lambda,\sigma}^{(m)}\,\varphi\right)(z) = \Lambda^{(m)}(\lambda,\sigma)\, p^{\lambda-\sigma} \int_S (1-s\overline{z})^{2\sigma,-2m}\, s^m\, \varphi(s) ds,$$

which is just (3.3). □

Theorem 3.3. *Introduce on* D *polar coordinates* $r,\ s:\ z=rs,\ 0\leqslant r\leqslant 1,\ s\in S$. *Let* $2\sigma\notin\mathbb{Z}$. *For any* K*-finite function* $\varphi\in\mathcal{D}(S)$, *the Poisson transform* $P_{\lambda,\sigma}^{(m)}\,\varphi$ *of* φ *has the following expansion in powers of* $p=1-r^2$:

$$\begin{aligned}
\left(P_{\lambda,\sigma}^{(m)}\,\varphi\right)(z) &= p^{-\lambda-\sigma-2}\, s^m \sum_{k=0}^{\infty} \left(C_{\sigma,k}^{(m)}\,\varphi\right)(s)\cdot p^k \\
&\quad + p^{\lambda+\sigma-1}\, s^m \sum_{k=0}^{\infty} \left(D_{\sigma,k}^{(m)}\,\varphi\right)(s)\,\cdot p^k.
\end{aligned} \tag{3.9}$$

Let us the factors $p^{\lambda-\sigma-2}$ and $p^{-\lambda+\sigma-1}$ in (3.9) *leading factors*. The factors yield that $P_{\lambda,\sigma}^{(m)}$ is meromorphic in σ, and has poles at the points

$$\sigma=\lambda-k, \quad \sigma=-\lambda-1+l \quad (k,l\in\mathbb{N}). \tag{3.10}$$

All poles are simple except in the case when the two sequences (3.10) have a non-empty intersection and the pole belongs to this intersection. This happens when $2\lambda+1\in\mathbb{N}$ and $0\leqslant k,\ \ l\leqslant 2\lambda+1$, $k+l=2\lambda+1$. In this case the pole μ is of the second order. Let us write down the principal part of the Laurent series of $P_{\lambda,\sigma}^{(m)}$ at the poles μ of the first and the second order respectively:

$$P_{\lambda,\sigma}^{(m)} = \frac{\widehat{P}_{\lambda,\mu}^{(m)}}{\sigma-\mu} + \cdots \tag{3.11}$$

$$P_{\lambda,\sigma}^{(m)} = \frac{\widehat{\widehat{P}}_{\lambda,\mu}^{(m)}}{(\sigma-\mu)^2} + \frac{\widehat{P}_{\lambda,\mu}^{(m)}}{\sigma-\mu} + \cdots \tag{3.12}$$

The first Laurent coefficient ($\widehat{P}_{\lambda,\mu}^{(m)}$ and $\widehat{\widehat{P}}_{\lambda,\mu}^{(m)}$ respectively) intertwines $T_{-\mu-1}$ with $R_{\lambda,m}$.

Let us write down the Laurent coefficients in (3.11) and (3.12) explicitly.

For that we introduce the following differential operators $W_{\sigma,k}^{(m)}$ on S. Let us set

$$V_{\sigma,m,n}(p) = (1-p)^{(m+n)/2} F(\sigma+1+m, \sigma+1+n; 2\sigma+2; p),$$

where F is the Gauss hypergeometric function. Expand V in powers of p:

$$V_{\sigma,m,n}(p) = \sum_{k=0}^{\infty} w_{\sigma,k}^{(m)}(n)\, p^k,$$

here $w_{\sigma,k}^{(m)}$ are polynomials in n of degree k. The coefficients of these polynomials are rational functions of σ with simple poles at $\sigma = -1, -3/2, \ldots, (-k-1)/2$. Now we set

$$W_{\sigma,k}^{(m)} = w_{\sigma,k}^{(m)} \left(\frac{1}{i} \frac{d}{d\alpha} \right).$$

If a pole μ belongs only to one of the sequences (3.10), then it is simple and

$$\widehat{P}_{\lambda,\lambda-k}^{(m)} = (-1)^{k+m} \frac{1}{k!}\, a_{-m}(\lambda-k)\, \xi_{\lambda,k}^{(m)}, \tag{3.13}$$

$$\widehat{P}_{\lambda,-\lambda-1+l}^{(m)} = (-1)^{l+m} \frac{1}{l!}\, \xi_{\lambda,l}^{(m)} \circ A_{\lambda-l}, \tag{3.14}$$

where $\xi_{\lambda,k}^{(m)}$ is the following operator $\mathcal{D}(S) \to \Sigma_k(\overline{D})$:

$$\xi_{\lambda,k}^{(m)} \varphi = s^m \sum_{n=0}^{k} (-1)^n \frac{k!}{(k-n)!} \left(W_{\lambda-k,n}^{(m)} \varphi \right)(s)\, \delta^{(k-n)}(p). \tag{3.15}$$

The operator $\xi_{\lambda,k}^{(m)}$ is meromorphic in λ. For fixed $k = 1, 2 \ldots$ it has poles (simple) at the points

$$\lambda = k-1,\ k-3/2,\ k-2, \ldots, \frac{k-1}{2}$$

(k poles in total). It intertwines $T_{-\lambda-1+k}$ with $L_{\lambda,m}$ (restricted to $\Sigma_k(\overline{D})$).

A number $\lambda_0 \in \mathbb{N}/2$ is a pole for $\xi_{\lambda,k}^{(m)}$ for those k that satisfy

$$\lambda_0 + 1 \leqslant k \leqslant 2\lambda_0 + 1.$$

In particular, let $\lambda_0 \in \mathbb{N}$. Denote by $\widehat{\xi}_{\lambda_0,k}^{(m)}$ the residue of $\xi_{\lambda,k}^{(m)}$ at $\lambda = \lambda_0$ and denote by $\widehat{W}_{\tau,k}^{(m)}$ the residue of $W_{\sigma,k}^{(m)}$ at the pole $\sigma = \tau$. The contribution to the residue of $\xi_{\lambda,k}^{(m)}$ is given by the summands in (3.15) for which $n \geqslant 2k - 2\lambda_0 - 1$. So we have (we omit the index 0) for $\lambda \in \mathbb{N}$ and $\lambda + 1 \leqslant k \leqslant 2\lambda + 1$:

$$\xi_{\lambda,k}^{(m)} \varphi = s^m \sum_{n=2k-2\lambda-1}^{k} (-1)^n \frac{k!}{(k-n)!} \left(\widehat{W}_{\lambda-k,n}^{(m)} \varphi \right)(s) \cdot \delta^{(k-n)}(p).$$

Let the pole μ belong to both sequences (3.10). This happens when $2\lambda + 1 \in \mathbb{N}$. Then $\mu = \lambda - k = -\lambda - 1 - l$, where $k, l \in \mathbb{N}$, so that $k + l = 2\lambda + 1$ and $l - k = 2\mu + 1$.

Let first $\lambda \in \mathbb{N}$. Then the pole μ is of the *second* order ($m \neq 0$). Here we have a difference with the case $m = 0$: in that case the pole μ was of the first order.

We shall write down, for $\lambda\in\mathbb{N}$, only the *first* Laurent coefficients $\widehat{\widehat{P}}$. The expressions for the residues are rather complicated and not interesting for us, even more because they turn out to be not concentrated at S.

If $\lambda+1\leqslant k\leqslant 2\lambda+1$ (so that $k>l$ and $\mu\leqslant -1$), then

$$\widehat{\widehat{P}}^{(m)}_{\lambda,\lambda-k}=(-1)^{k+m}\,\frac{1}{k!}\,a_{-m}(\lambda-k)\,\widehat{\xi}^{(m)}_{\lambda,k},$$

and if $\lambda+1\leqslant l\leqslant 2k+1$ (so that $k<l$ and $\mu\geqslant 0$), then

$$\widehat{\widehat{P}}^{(m)}_{\lambda,-\lambda-1+l}=(-1)^{l+m}\,\frac{1}{l!}\,\widehat{\xi}^{(m)}_{\lambda,l}\circ A_{\lambda-l}.$$

Therefore, the operator $\widehat{\xi}^{(m)}_{\lambda,k}$ intertwines $T_{-\lambda-1+k}$ with $L_{\lambda,m}$ restricted to $\Sigma_k(\overline{D})$.

Let now $\lambda\in -1/2+\mathbb{N}$. This case is similar to such a case for $m=0$. The pole μ is of the second order.

If $k\leqslant l$, then

$$\begin{aligned}\widehat{\widehat{P}}^{(m)}_{\lambda,\mu}&=2\,\frac{(-1)^{k+m}}{k!}\,\widehat{a}_{-m}(\mu)\,\xi^{(m)}_{\lambda,k},\\ \widehat{P}^{(m)}_{\lambda,\mu}\,\varphi&=-s^m\sum_{n=0}^{l}\frac{(-1)^{l-n}}{(l-n)!}\,\widetilde{C}^{(m)}_{\mu,n}\varphi\cdot\delta^{l-n}(p),\end{aligned}$$

and if $k\geqslant l$, then

$$\begin{aligned}\widehat{\widehat{P}}^{(m)}_{\lambda,\mu}&=2\,\frac{(-1)^{l+m}}{l!}\,\xi^{(m)}_{\lambda,l}\circ\widehat{A}_{\lambda-l},\\ \widehat{P}^{(m)}_{\lambda,\mu}\,\varphi&=s^m\sum_{n=0}^{k}\frac{(-1)^{k-n}}{(k-n)!}\,\widetilde{D}^{(m)}_{\mu,n}\varphi\cdot\delta^{(k-n)}(p),\end{aligned}$$

where $\widehat{a}_{-m}(\mu)$ is the residue of $a_{-m}(\sigma)$ at $\sigma=\mu$ ($\widehat{A}_\tau$ is the residue of A_σ at $\sigma=\tau$) and

$$\widetilde{C}^{(m)}_{\mu,n}=\begin{cases}C^{(m)}_{\mu,n}, & n<2\mu+1,\\ C^{0}{}^{(m)}_{\mu,n}-D^{0}{}^{(m)}_{\mu,n-2\mu-1}, & n\geqslant 2\mu+1,\end{cases}$$

$$\widetilde{D}^{(m)}_{\mu,}=\begin{cases}D^{(m)}_{\mu,n}, & n<-2\mu-1,\\ D^{0}{}^{(m)}_{\mu,n}-C^{0}{}^{(m)}_{\mu,n+2\mu+1}, & n\geqslant -2\mu-1.\end{cases}$$

Theorem 3.4. *Up to a factor, the composition of the operators* $Q_{\lambda,m}$ *and* $\xi^{(m)}_{\lambda,k}$ *is the Poisson transform* $P^{(m)}_{-\lambda-2,\lambda-k}$*:*

$$Q_{\lambda,m}\,\xi^{(m)}_{\lambda,k}=q^{(m)}_{\lambda,k}\cdot P^{(m)}_{-\lambda-2,\lambda-k},\tag{3.16}$$

where

$$q^{(m)}_{\lambda,k}=\frac{1}{2}\,(-1)^{k+m}\,k!\,a_{-m}(-\lambda-1+k)\,\Lambda^{(m)}_k(\lambda),\tag{3.17}$$

$$\Lambda^{(m)}_k(\lambda)=-\frac{1}{2\pi^2}\,(2\lambda-2k+1)\,\frac{\Gamma(\lambda+m+1)\,\Gamma(\lambda-m+2)}{k!\,\Gamma(2\lambda+2-k)}.$$

Proof. Taking the residue of both sides of (3.3) at $\sigma=\lambda-k$ and using (3.13), we obtain (3.16), where

$$q^{(m)}_{\lambda,k}=(-1)^{k+m}\,k!\,\frac{1}{a_{-m}(\lambda-k)}\,\mathrm{Res}_{\sigma=\lambda-k}\,\Lambda^{(m)}(\lambda,\sigma).$$

The latter residue is equal to

$$\frac{\pi}{2\lambda - 2k + 1}\,\mathrm{tg}\,\lambda\pi \cdot \Lambda_k^{(m)}(\lambda).$$

Finally, computing the product $a_{-m}(\sigma)\, a_{-m}(-\sigma - 1)$, we obtain expression (3.17). □

Remark. Formula (3.3) seems to contain a contradiction: indeed, the Poisson transform $P^{(m)}_{-\lambda-2,\sigma}$ in the right hand side has poles at the points $\sigma = \lambda + 1 + l$ and $\sigma = -\lambda - 2 - k$ $(k, l \in \mathbb{N})$, but the left hand side seems to have no poles at these points. In fact, the left hand side does have poles at these points; the poles in question are poles of *distributions*; the left hand side, regarded as a distribution, assigns to a function $f \in \mathcal{D}(\overline{D})$ the scalar

$$\langle Q_{\lambda,m}\, P^{(m)}_{\lambda,\sigma},\, f\rangle_D = \langle P^{(m)}_{\lambda,\sigma},\, Q_{\overline{\lambda},m} f\rangle_D,$$

but the function $Q_{\lambda,m}\overline{f}$ has asymptotics $C_1 + C_2\, p^{2\lambda+2}$ when $p \to 0$, and the function $p^{2\lambda+2}$ together with the leading terms $p^{-\lambda-\sigma-2}$ and $p^{-\lambda+\sigma-1}$ of $P^{(m)}_{\lambda,\sigma}$ gives the desired poles.

§ 4. Fourier transform

Let $\lambda, \sigma \in \mathbb{C}$ and $m \in \mathbb{Z}$. We define the *Fourier transform associated with the canonical representation* $R_{\lambda,m}$ as the map $F^{(m)}_{\lambda,\sigma} : \mathcal{D}(\overline{D}) \to \mathcal{D}(S)$ by the following formula

$$\left(F^{(m)}_{\lambda,\sigma} f\right)(s) = s^{-m} \int_D (1 - z\overline{z})^{2\sigma, 2m}\, p^{\lambda-\sigma}\, f(z)dxdy.$$

The integral converges absolutely for $\mathrm{Re}\,(\lambda - \sigma) > -1$, $\mathrm{Re}\,(\lambda + \sigma) > -2$ and can be meromorphically continued in σ and λ.

Theorem 4.1. T*he Poisson and the Fourier transform are conjugate to each other:*

$$\langle F^{(m)}_{\lambda,\sigma} f,\, \varphi\rangle_S = \langle f,\, P^{(m)}_{-\overline{\lambda}-2,\overline{\sigma}}\, \varphi\rangle_D.$$

This allows to transfer statements about the Poisson transform to the Fourier transform.

The Fourier transform interacts with the intertwining operators as follows:

$$\begin{aligned} A_\sigma\, F^{(m)}_{\lambda,\sigma} &= a_{-m}(\sigma)\, F^{(m)}_{\lambda,-\sigma-1}, \\ F^{(m)}_{-\lambda-2,\sigma}\, Q_{\lambda,m} &= \Lambda^{(m)}(\lambda,\sigma)\, F^{(m)}_{\lambda,\sigma}. \end{aligned}$$

It has poles in σ at the points

$$\sigma = -\lambda - 2 - k, \quad \sigma = \lambda + 1 + l \quad (k, l \in \mathbb{N}). \tag{4.1}$$

All poles are simple, except the case $-2\lambda - 3 \in \mathbb{N}$ and the pole μ belongs to both sequences (4.1), i.e $0 \leqslant k, l \leqslant -2\lambda - 3$ and $k + l = -2\lambda - 3$. In this case μ is of the second order.

For the Laurent coefficients of the Fourier transform we use a similar notation as in case of the Poisson transform.

The first Laurent coefficient (i.e. $\widehat{F}^{(m)}_{\lambda,\mu}$ if μ is of the first order and $\widehat{\widehat{F}}^{(m)}_{\lambda,\mu}$ if μ is of the second order) intertwines $R_{\lambda,m}$ with T_μ).

Let us write down $\widehat{F}^{(m)}_{\lambda,\mu}$ and $\widehat{\widehat{F}}^{(m)}_{\lambda,\mu}$ explicitly.

If the pole μ belongs to one of the sequences (4.1), then it is simple and

$$\begin{aligned} \widehat{F}^{(m)}_{\lambda,-\lambda-2-k} &= \frac{1}{2}\,(-1)^m\, a_{-m}(-\lambda - 2 - k)\, b^{(m)}_{\lambda,k}, \\ \widehat{F}^{(m)}_{\lambda,\lambda+1+l} &= -\frac{1}{2}\,(-1)^m\, A_{-\lambda-2-l}\, b^{(m)}_{\lambda,l}, \end{aligned}$$

where $b^{(m)}_{\lambda,k}$ is a "boundary" operator $\mathcal{D}(\overline{D}) \to \mathcal{D}(S)$ which is defined in terms of the Taylor coefficients c_n of f as follows:

$$b^{(m)}_{\lambda,k}(f) = \sum_{n=0}^{k} W^{(m)}_{-\lambda-2-k,k-n}\,(s^{-m}c_n).$$

theorem 4.1 now gives:

Theorem 4.2. *The operators $\xi^{(m)}$ and $b^{(m)}$ are conjugate to each other (up to a factor):*

$$\langle f,\, \xi^{(m)}_{-\overline{\lambda}-2,k}\,\varphi\rangle_D = \frac{1}{2}\,(-1)^k\, k!\,\langle b^{(m)}_{\lambda,k}(f),\, \varphi\rangle_S.$$

The operator $b^{(m)}_{\lambda,k}$ intertwines $R_{\lambda,m}$ with $T_{-\lambda-2-k}$. It is meromorphic in λ. For fixed $k=1,2,\ldots$ it has poles (simple) at the points:

$$\lambda = -k-1, -k-1/2, \ldots, \frac{-k-3}{2}$$

(k poles in total). A scalar $\lambda_0 \in -2-\mathbb{N}/2$ is a pole for $b^{(m)}_{\lambda,k}$ if

$$-\lambda_0 - 1 \leqslant k \leqslant -2\lambda_0 - 3.$$

In particular, let $\lambda_0 \in -2-\mathbb{N}$. Denote by $\widehat{b}^{(m)}_{\lambda_0,k}$ the residue of $b^{(m)}_{\lambda,k}$ at $\lambda=\lambda_0$. Then (cf. § 3) (we omit the index 0) for $\lambda \in -2-\mathbb{N}$ and $-\lambda-1 \leqslant k \leqslant -2\lambda-3$ we have

$$\widehat{b}^{(m)}_{\lambda,k}(f) = -\sum_{n=0}^{-2\lambda-3-k} \widehat{W}^{(m)}_{-\lambda-2-k,k-n}\,(s^{-m}c_n).$$

Let the pole μ belong th both sequences (4.1). This happens when $-2\lambda-3 \in \mathbb{N}$. Then $\mu=-\lambda-{}$ $-2-k=k+1+l$, where $k,l \in \mathbb{N}$, so that $k+l=-2\lambda-3$, $l-k=2\mu+1$. This pole is of the second order.

Let $\mu \in -2-\mathbb{N}$. As in § 3 we write only down the first Laurent coefficient: if $-\lambda-1 \leqslant k$ (then $k>l$ and $\mu \leqslant -1$), then

$$\widehat{\widehat{F}}{}^{(m)}_{\lambda,-\lambda-2-k} = \frac{1}{2}\,(-1)^m\, a_{-m}(-\lambda-2-k)\,\widehat{b}^{(m)}_{\lambda,k},$$

and if $-\lambda-1 \leqslant l$ (then $k<l$ and $\mu \geqslant 0$), then

$$\widehat{\widehat{F}}{}^{(m)}_{\lambda,\lambda+1+l} = -\frac{1}{2}\,(-1)^m\, A_{-\lambda-2-l}\,\widehat{b}^{(m)}_{\lambda,l}.$$

Let $\mu \in -5/2-\mathbb{N}$. If $k \leqslant l$, then

$$\begin{aligned} \widehat{\widehat{F}}{}^{(m)}_{\lambda,\mu} &= (-1)^m\, \widehat{a}_{-m}(\mu)\, b^{(m)}_{\lambda,k} \\ \widehat{F}^{(m)}_{\lambda,\mu} f &= -\frac{1}{2}\sum_{n=0}^{l} \widetilde{C}^{(m)}_{\mu,n}\,(s^{-m}c_{l-n}), \end{aligned}$$

and if $k \geqslant l$, then

$$\begin{aligned} \widehat{\widehat{F}}{}^{(m)}_{\lambda,\mu} &= (-1)^m\, \widehat{A}_{-\lambda-2-l}\, b^{(m)}_{\lambda,l}, \\ \widehat{F}^{(m)}_{\lambda,\mu} f &= \frac{1}{2}\sum_{n=0}^{k} \widetilde{D}^{(m)}_{\mu,n}\,(s^{-m}\, c_{k-n}). \end{aligned}$$

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Received 4 September 2017

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УДК 517.98
DOI: 10.20310/1810-0198-2017-22-6-1218-1228

КАНОНИЧЕСКИЕ ПРЕДСТАВЛЕНИЯ НА ПЛОСКОСТИ ЛОБАЧЕВСКОГО В СЕЧЕНИЯХ ЛИНЕЙНЫХ РАССЛОЕНИЙ

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Мы описываем канонические представления, связанные с сечениями линейных расслоений, соответствующие граничные представления и преобразования Пуассона и Фурье.
Ключевые слова: плоскость Лобачевского; канонические представления; обобщенные функции; граничные представления; преобразования Пуассона и Фурье

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Поступила в редакцию 4 сентября 2017 г.

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For citation: Grosheva L.I. Canonical and boundary representations on the Lobachevsky plane associated with linear bundles. *Vestnik Tambovskogo universiteta. Seriya Estestvennye i tekhnicheskie nauki – Tambov University Reports. Series: Natural and Technical Sciences*, 2017, vol. 22, no. 6, pp. 1218–1228. DOI: 10.20310/1810-0198-2017-22-6-1218-1228 (In Engl., Abstr. in Russian).

Для цитирования: *Грошева Л.И.* Канонические представления на плоскости Лобачевского в сечениях линейных расслоений // Вестник Тамбовского университета. Серия Естественные и технические науки. Тамбов, 2017. Т. 22. Вып. 6. С. 1218–1228. DOI: 10.20310/1810-0198-2017-22-6-1218-1228.