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VOLTERRA OPERATOR INCLUSIONS IN THE THEORY OF GENERALIZED NEURAL FIELD MODELS WITH CONTROL. II

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We obtained conditions for solvability of Volterra operator inclusions and continuous dependence of the solutions on a parameter. These results were implemented to investigation of generalized neural field equations involving control.
Key words: Volterra operator inclusions; neural field equations; control; existence of solutions; continuous dependence on parameters

3. Integro-differential inclusion with parameter.

In the first part of the present paper (see [1]), we introduced a controllable integro-differential equation and posed the corresponding well-posedness problem. The form of the integro-differential equation introduced describes a broad class of equations used in modeling of electrical activity in the brain cortex (see, e.g. [2], [3]). The particular type of solutions, namely, spatially localized solutions, which we have focused on in the first part of the paper, characterizes the electrical activity of the brain prevalent during it's normal functioning (see, e.g. [4], [5]). Electrical stimulation of the brain using e.g. in therapy of Epilepsy and Parkinson's disease, following the ideas of [6]–[10], was considered as control. The control problem obtained was represented in the form of integro-differential inclusion in the following way

$$w(t,x) \in \int_a^t \int_{R^m} f(t,s,x,y,w(s,y),U,\lambda)dyds, \ t \in [a,\infty) \ , x \in R^m, \ \lambda \in \Lambda, \tag{3.1}$$

$$\lim_{|x|\to\infty} w(t,x) = 0, \ t \in [a,\infty). \tag{3.2}$$

Using the results from Section 2 of the previous part of the paper (see [1]), we investigate here the solvability of the problem above and continuous dependence of the solutions obtained on the parameter λ from some metric space Λ. Here U is some compact subset of R^k.

Throughout the section, we will use the following notation. Let $L(\Omega,\mu,R^n)$ be the space of all measurable and integrable functions $\chi:\Omega\to R^n$ with the norm $\|\chi\|_{L(\Omega,\mu,R^n)} = \int\limits_{\Omega} |\chi(s)|ds$; $C_0(\Omega,R^n)$

be the space of all continuous functions $\vartheta : \Omega \to R^n$ satisfying the additional condition $\lim\limits_{|x|\to\infty} \vartheta(x) = 0$ in the case if Ω is unbounded, with the norm $\|\vartheta\|_{C_0(\Omega,R^n)} = \max\limits_{x\in\Omega} |\vartheta(x)|$; and $C([a,b], C_0(\Omega, R^n))$ be the space of all continuous functions $\nu : [a,b] \to C_0(\Omega, R^n)$, with the norm $\|\nu\|_{C([a,b],C_0(\Omega,R^n))} = \max\limits_{t\in[a,b]} \|\nu(t)\|_{C_0(\Omega,R^n)}$.

We assume that for some $\lambda_0 \in \Lambda$ and $r_0 > 0$, the following conditions are fulfilled:

(i) For any $t \in [a,\infty)$, $w \in R^n$, $x \in R^m$ and any $u \in U$, $\lambda \in \Lambda$, the function $f(t,\cdot,x,\cdot,w,u,\lambda) : [a,\infty) \times R^m \to R^n$ is measurable.

(ii) For almost all $(s,y) \in [a,\infty) \times R^m$ and any $\lambda \in \Lambda$, the function $f(\cdot,s,\cdot,y,\cdot,\cdot,\lambda)$ is continuous.

(iii) For any $b \in (a,\infty)$ and any $r > 0$, it holds true that

$$\lim_{\mathfrak{r}\to\infty} \sup_{t\in[a,b],x\in R^m\setminus B_{R^m}(0,\mathfrak{r})} \Big| \int\limits_a^t \int\limits_{R^m} f(t,s,x,y,w,u,\lambda_0+\Delta)dyds \Big| = 0$$

for any $w \in B_{R^n}(0,r)$, uniformly for all $u \in U$ and $\Delta \in B_\Lambda(0,r_0)$.

(iv) For any $b \in (a,\infty)$ and any $r > 0$, there exists such $g_{(b,r)} \in L([a,b] \times R^m, \mu, R)$ that

$$|f(t,s,x,y,w,u,\lambda)| \le g_{(b,r)}(s,y)$$

for all $x \in R^m$, $w \in B_{R^n}(0,r)$, $t \in [a,b]$, $u \in U$, $\lambda \in B_\Lambda(\lambda_0,r_0)$ and almost all $(s,y) \in [a,b] \times R^m$.

D e f i n i t i o n 3. 1. Choose an arbitrary $u \in B_M(u_0,r_0)$. We define a *local solution* to the problem (3.1) – (3.2), defined on $[a,a+\gamma]$, $\gamma \in (0,\infty)$, to be a function $w_\gamma \in C([a,a+\gamma], C_0(R^m,R^n))$, that satisfies the equation (3.1) on $[a,a+\gamma]$. We define a *maximally extended solution* to the problem (3.1)–(3.2), defined on $[a,a+\eta)$, $\eta \in (0,\infty)$, to be a map $w_\eta : [a,a+\eta) \to C_0(R^m,R^n)$, whose restriction w_γ on $[a,a+\gamma]$ with any $\gamma < \eta$ is its local solution and $\lim\limits_{\gamma\to\eta-0} \|w_\gamma\|_{C([a,a+\gamma],C_0(R^m,R^n))} = \infty$. We define an ∞ *-solution* to the problem (3.1)–(3.2) to be a map $w : [a,\infty) \to C_0(R^m,R^n)$, whose restriction w_γ on $[a,a+\gamma]$ for any $\gamma \in (0,\infty)$ is its local solution.

For any $\lambda \in \Lambda$ and $\gamma \in (0,1)$, we denote by $\mathbb{S}_\gamma(\lambda)$ the set of $\upsilon(\gamma)$ -local solutions to (3.1)–(3.2) corresponding to $\lambda \in \Lambda$.

T h e o r e m 3. 1. *Let the assumptions (i) – (iv) hold true. Assume that the following conditions are satisfied:*

1) *For the given $r_0 > 0$ and any $r > 0$ there exists $\widetilde{f}_r(s,y) \in L([a,\infty) \times R^m, \mu, \mathbb{R})$ such that for which*

$$\sup_{u^1\in U} \inf_{u^2\in U} |f(t,s,x,y,w^1,u^1,\lambda) - f(t,s,x,y,w^2,u^2,\lambda)| \le \widetilde{f}_r(s,y)|w^1 - w^2|$$

for all $w^1, w^2 \in B_{R^n}(0,r)$, $u \in \mathfrak{U}$, $\lambda \in B_\Lambda(\lambda_0,r_0)$, $t \in [a,\infty)$, $x \in B_{R^m}(0,r)$.

2) *For any $w \in R^n$, $t \in [a,\infty)$, $x \in R^m$, and any sequence $\lambda_i \to \lambda_0$, it holds true that*

$$\sup_{\widehat{u}\in U} \inf_{u\in U} |f(t,\cdot,x,\cdot,w,\widehat{u},\lambda_0) - f(t,\cdot,x,\cdot,w,u,\lambda_i)| \to 0$$

in measure on $[a,\infty) \times R^m$.

Then for each $(u,\lambda) \in U \times B_\Lambda(\lambda_0,r_0)$, the problem (3.1)–(3.2) has a local solution, and each local solution can be extended to an ∞ -solution or maximally extended solution. Moreover, if at

$\lambda=\lambda_0$ *the problem* (3.1)–(3.2) *has a local solution* $w_{0\gamma}$ *defined on* $[a,a+\gamma]$, *then for any* λ *sufficiently close to* λ_0, *the problem* (3.1)–(3.2) *has a local solution* $w_\gamma=w_\gamma(\lambda)$ *defined on* $[a,a+\gamma]$ *and the set-valued mapping* $\lambda\mapsto\mathbb{S}_\gamma(\lambda)$ *is lower semi-continuous at* λ_0.

P r o o f.

From the conditions (i) – (iv), one can observe that:

a) for any $(u,\lambda)\in U\times B_\Lambda(\lambda_0,r_0)$ and each $b>a$,

$$\int\limits_a^{(\cdot)}\int\limits_{R^m} f(\cdot,s,\cdot,y,w(s,y),u,\lambda)d\nu(y)ds\in C([a,b],C_0(R^m,R^n));$$

b) $F(\cdot,\lambda):C([a,b],C_0(R^m,R^n))\to clos(C([a,b],C_0(R^m,R^n)))$,

$$F(w,\lambda)=\int\limits_a^{(\cdot)}\int\limits_{R^m} f(\cdot,s,\cdot,y,w(s,y),U,\lambda)d\nu(y)ds,$$

for all $\lambda\in B_\Lambda(\lambda_0,r_0)$ due to the theorem 1.2.29 (see [11]), continuity of F in the variable w and regularity of the space $C([a,b],C_0(R^m,R^n))$.

We are going to check now the conditions of Theorem 2.2 from the previous section.

Choose an arbitrary $b\in(a,\infty)$, $q_0<1$, $r>0$. Let $\gamma\in(0,b-a)$ and $w_1(t,\cdot)=w_2(t,\cdot)$, $t\in[a,a+\gamma]$, where $w_1,w_2\in B_{C([a,b],C_0(R^m,R^n))}(0,r)$. Using assumptions (i)–(iv) and condition 1) of Theorem 3.1, we get the following estimates

$$\sup_{t\in[a,a+\gamma+\delta],x\in R^m}\Big|\int\limits_a^t\int\limits_{R^m} f(t,s,x,y,w^1(s,y),u,\lambda)d\nu(y)ds-$$

$$-\int\limits_a^t\int\limits_{R^m} f(t,s,x,y,w^2(s,y),u,\lambda)d\nu(y)ds\Big|\le$$

$$\varepsilon/2+\sup_{t\in[a,a+\gamma+\delta],x\in B_{R^m}(0,r_\varepsilon)}\int\limits_{a+\gamma}^{a+\gamma+\delta}\int\limits_{R^m}\Big|f(t,s,x,y,w^1(s,y),u,\lambda)d\nu(y)ds-$$

$$-f(t,s,x,y,w^2(s,y),u,\lambda)\Big|d\nu(y)ds\le$$

$$\varepsilon/2+\sup_{t\in[a,a+\gamma+\delta],x\in B_{R^m}(0,r_\varepsilon)}\Big|\int\limits_{a+\gamma}^{a+\gamma+\delta}\int\limits_{R^m}\widetilde{f}_r(s,y)\|w^1-w^2\|_{C([a,b],BC(R^m,R^n))}d\nu(y)ds\Big|\le\varepsilon$$

for all $(u,\lambda)\in U\times B_\Lambda(\lambda_0,r_0)$. Here, $r_\varepsilon>0$, $\delta>0$ can be chosen in a such way that $\varepsilon<q_0$. The estimate above yields that

$$h_{C([a,a+\gamma+\delta],BC(R^m,R^n))}(F_{\gamma+\delta}\overline{w}^1_{\gamma+\delta},F_{\gamma+\delta}\overline{w}^2_{\gamma+\delta})\le q_0 d_{C([a,a+\gamma+\delta],BC(R^m,R^n))}(\overline{w}^1_{\gamma+\delta},\overline{w}^2_{\gamma+\delta}).$$

Thus, we checked that condition $\mathfrak{q}_2)$ is satisfied. The verification of condition $\mathfrak{q}_1)$ is analogous.

From the proof of Theorem 2.1 (see [1]), we can deduce that, if for each $b>a$ there exist at least one local solution defined on $[a,b]$ with bounded norm, we can obtain an ∞-solution. Otherwise, we get a maximally extended solution.

Next, we check the condition 2) of Theorem 2.2. from [1]. In order to do that, we take arbitrary $\varepsilon>0$, $\widehat{w}\in C([a,b],C_0(R^m,R^n))$, $w_i\subset C([a,b],C_0(R^m,R^n))$, $\{\lambda_i\}\subset\Lambda$, $\|\widehat{w}-w_i\|_{C([a,b],C_0(R^m,R^n))}$, $u_i\to u_0$ ($i\to\infty$), and estimate

$$h_{C([a,b],C_0(R^m,R^n))}(F(\widehat{w},u_0)-F(w_i,u_i))=$$

$$\sup_{\widehat{u}\in U,t\in[a,b],x\in R^m}\inf_{u\in U}\Big|\int\limits_a^t\int\limits_{R^m}f(t,s,x,y,\widehat{w}(s,y),\widehat{u},\lambda_0)d\nu(y)ds-$$

$$-\int\limits_a^t\int\limits_{R^m}f(t,s,x,y,w_i(s,y),u,\lambda_i)d\nu(y)ds\Big|\le$$

$$\le\varepsilon/3+\sup_{\widehat{u}\in U,t\in[a,b],x\in B_{R^m}(0,r_\varepsilon)}\inf_{u\in U}\Big|\int\limits_a^t\int\limits_{R^m}f(t,s,x,y,\widehat{w}(s,y),\widehat{u},\lambda_0)d\nu(y)ds-$$

$$-\int\limits_a^t\int\limits_{R^m}f(t,s,x,y,w_i(s,y),u(s,y),\lambda_i)d\nu(y)ds\Big|=$$

$$=\varepsilon/3+\sup_{\widehat{u}\in U,t\in[a,b],x\in B_{R^m}(0,r_\varepsilon)}\inf_{u\in U}\int\limits_a^t\int\limits_{R^m}\Big(\big|f(t,s,x,y,\widehat{w}(s,y),\widehat{u},\lambda_0)-f(t,s,x,y,\widehat{w}(s,y),u,\lambda_i)\big|+$$

$$+\big|f(t,s,x,y,\widehat{w}(s,y),u,\lambda_i)-f(t,s,x,y,w_i(s,y),u(s,y),\lambda_i)\big|\Big)d\nu(y)ds.$$

Estimating the first summand of the integrand, we get

$$\sup_{\widehat{u}\in U}\inf_{u\in U}\big|f(t,s,x,y,\widehat{w}(s,y),\widehat{u},\lambda_0)-f(t,s,x,y,\widehat{w}(s,y),u,\lambda_i)\big|\le$$

$$\le\sup_{\widehat{u}\in U}\inf_{u\in U}\big|f(t,s,x,y,\widehat{w}(s,y),\widehat{u},\lambda_0)-f(\bar{t^\varepsilon},s,\bar{x^\varepsilon},y,\overline{\widehat{w}^\varepsilon},\widehat{u}(s,y),\lambda_0)|+$$

$$+\sup_{\widehat{u}\in U}\inf_{u\in U}\big|f(\bar{t^\varepsilon},s,\bar{x^\varepsilon},y,\overline{\widehat{w}^\varepsilon},\widehat{u}(s,y),\lambda_0)-f(\bar{t^\varepsilon},s,\bar{x^\varepsilon},y,\overline{\widehat{w}^\varepsilon},u,\lambda_i)\big|+$$

$$+\sup_{\widehat{u}\in U}\inf_{u\in U}\big|f(\bar{t^\varepsilon},s,\bar{x^\varepsilon},y,\overline{\widehat{w}^\varepsilon},u,\lambda_i)-f(t,s,x,y,\widehat{w}(s,y),u(s,y),\lambda_i)\big|.$$

Here $\bar{t^\varepsilon}$, $\bar{x^\varepsilon}$, $\overline{\widehat{w}^\varepsilon}$ are approximations of t, x, $\widehat{w}(s,y)$, taking finite number of values (on their compact ranges of definition). Thus, using the condition 2) of Theorem 3.1 and the assumption (ii), the first and the third summands on the right-hand side of the inequality go to 0 uniformly with respect to $(s,y)\in[a,b]\times R^m$ and the second summand go to 0 in measure on $[a,b]\times R^m$ with respect to the variables s and y.

Next, estimation of $\big|f(t,s,x,y,\widehat{w}(s,y),u,\lambda_i)-f(t,s,x,y,w_i(s,y),u,\lambda_i)\big|$ using the assumption (ii), gives uniform convergence of this expression to 0 on $[a,b]\times R^m$.

Thus, we can find such I that for any $i>I$, we get

$$\sup_{t\in[a,b],x\in B_{R^m}(0,r_\varepsilon)}\int\limits_a^t\int\limits_{R^m}\Big(\big|f(t,s,x,y,\widehat{w}(s,y),u_0)-f(t,s,x,y,\widehat{w}(s,y),u_i)\big|+$$

$$+\big|f(t,s,x,y,\widehat{w}(s,y),u_i)-f(t,s,x,y,w_i(s,y),u_i)\big|\Big)d\nu(y)ds\le2\varepsilon/3,$$

which concludes the verification of Theorem 2.1 conditions and completes the proof.

□

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ОПЕРАТОРНЫЕ ВКЛЮЧЕНИЯ ВОЛЬТЕРРЫ В ОБОБЩЕННЫХ МОДЕЛЯХ НЕЙРОПОЛЕЙ С УПРАВЛЕНИЕМ. II

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Получены условия разрешимости операторных включений Вольтерры и непрерывной зависимости решений от параметра. Результаты применены к исследованию обобщенных моделей нейрополей с управлением.
Ключевые слова: операторные включения Вольтерры; модели нейрополей; управление; существование решений; непрерывная зависимость от параметров

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