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Asymptotics for the Radon transform on hyperbolic spaces

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Асимптотика преобразования Радона на гиперболических пространствах

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Abstract. Let G/H be a hyperbolic space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let K be a maximal compact subgroup of G. Let D denote a certain explicit invariant differential operator, such that the non-cuspidal discrete series belong to the kernel of D. For any L^2-Schwartz function f on G/H, we prove that the Abel transform $\mathcal{A}(Df)$ of Df is a Schwartz function. This is an extension of a result established in [2] for K-finite and $K\cap H$-invariant functions.

Keywords: hyperbolic spaces; Radon transform; cuspidal discrete series; Abel transform

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Аннотация. Пусть G/H — гиперболическое пространство над $\mathbb{R}$, $\mathbb{C}$ или $\mathbb{H}$, пусть K — максимальная компактная подгруппа группы G. Пусть D обозначает некоторый явно выписываемый дифференциальный оператор — такой, что некаспидальные дискретные серии принадлежат ядру оператора D. Мы доказываем, что для всякой функции f из пространства L^2-Шварца на G/H преобразование Абеля $\mathcal{A}(Df)$ функции Df есть функция Шварца. Это — расширение результата, установленного в [2] для K-финитных и $K \cap H$-инвариантных функций.

Ключевые слова: гиперболические пространства; преобразование Радона; каспидальные дискретные серии; преобразование Абеля

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§ 1. Introduction

The *Radon transform* R on the hyperbolic spaces G/H,

$$Rf = \int_{N^*} f(\cdot\, nH)\, dn,$$

where $N^* \subset G$ is a certain unipotent subgroup, and the associated *Abel transform* $\mathcal{A}$, were introduced and studied in [1] and [2]. Generalizing Harish-Chandra's notion of cusp forms for real semisimple Lie groups, a discrete series is said to be *cuspidal* if it is annihilated by the Radon transform. In contrast with the Lie group case, however, *non-cuspidal* discrete series exist. For the projective hyperbolic spaces, these are precisely the spherical discrete series, but for some real non-projective hyperbolic spaces, there also exist non-spherical non-cuspidal discrete series.

Let $\mathcal{C}^2(G/H)$ denote the space of L^2-Schwartz functions on G/H. Except for some boundary cases, $\mathcal{A}$ maps $\mathcal{C}^2(G/H)$ into Schwartz functions in the absence of non-cuspidal discrete series. On the other hand, $\mathcal{A}f$ can be explicitly calculated for functions f belonging to the non-cuspidal discrete series. To complete the picture, we prove below that $\mathcal{A}$ essentially maps the orthocomplement in $\mathcal{C}^2(G/H)$ of the non-cuspidal discrete series into Schwartz functions. To be more precise, let $\Delta_\rho = \Delta + \rho_\mathfrak{q}^2$, where Δ denotes the Laplace–Beltrami operator on G/H, and consider the G-invariant differential operator $D = \Delta_\rho(\Delta_\rho - \lambda_1^2)\ldots(\Delta_\rho - \lambda_r^2)$, where $\lambda_1, \ldots, \lambda_r$ are the parameters of the non-cuspidal discrete series. Then $\mathcal{A}(Df)$ is a Schwartz function. This extends our previous result, [2, Theorem 6.1], valid only for the dense G-invariant subspace of $\mathcal{C}^2(G/H)$ generated by the K-irreducible $(K \cap H)$-invariant functions, to all Schwartz functions.

In [2] we also considered the exceptional case corresponding to the Cayley numbers $\mathbb{O}$. We expect our new result to hold for this case as well, but we have not been through the rather cumbersome details.

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§ 2. The Radon transform

In this section, we define the Radon transform and the Abel transform for the projective hyperbolic spaces over the classical fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. We have tried to keep the presentation and notation to a minimum, see [1] and [2] for further details (including results and proofs).

Let $x \mapsto \overline{x}$ be the standard (anti-) involution of $\mathbb{F}$. Let $p \geqslant 0$, $q \geqslant 1$ be two integers, and consider the Hermitian form $[\cdot, \cdot]$ on $\mathbb{F}^{p+q+2}$ given by

$$[x, y] = x_1\overline{y}_1 + \ldots + x_{p+1}\overline{y}_{p+1} - x_{p+2}\overline{y}_{p+2} - \ldots - x_{p+1+q+1}\overline{y}_{p+1+q+1}\,,$$

where $x, y \in \mathbb{F}^{p+q+2}$. Let $G = \mathrm{U}\,(p+1, q+1; \mathbb{F})$ denote the group of $(p+q+2)\times(p+q+2)$ matrices over $\mathbb{F}$ preserving $[\cdot\,,\,\cdot]$. Thus $\mathrm{U}\,(p+1, q+1; \mathbb{R}) = \mathrm{O}(p+1, q+1)$, $\mathrm{U}\,(p+1, q+1; \mathbb{C}) = \mathrm{U}\,(p+1, q+1)$ and $\mathrm{U}\,(p+1, q+1; \mathbb{H}) = \mathrm{Sp}\,(p+1, q+1)$ in standard notation. Put $\mathrm{U}\,(p\,; \mathbb{F}) = \mathrm{U}\,(p, 0; \mathbb{F})$, and let $K = \mathrm{U}\,(p+1; \mathbb{F}) \times \mathrm{U}\,(q+1; \mathbb{F})$ be the maximal compact subgroup of G fixed by the Cartan involution on G.

Let $x_0 = (0, \ldots, 0, 1)^T$, where superscript T indicates transpose. Let H be the subgroup $\mathrm{U}\,(p+1, q; \mathbb{F}) \times \mathrm{U}\,(1; \mathbb{F})$ of G stabilizing the line $\mathbb{F}\cdot x_0$ in $\mathbb{F}^{p+q+2}$. The reductive symmetric space G/H can be identified with the projective hyperbolic space $\mathbb{X} = \mathbb{X}(p+1, q+1; \mathbb{F})$,

$$\mathbb{X} = \{z \in \mathbb{F}^{p+q+2} : [z, z] = -1\}/\sim,$$

where $\sim$ is the equivalence relation $z \sim zu$, $u \in \mathbb{F}^*$.

Let X_t, for $t \in \mathbb{R}$, denote the following element in the Lie algebra $\mathfrak{g}$ of G:

$$X_t = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

(a matrix of order $p+q+2$). Let $\mathfrak{a}_\mathfrak{q}$ denote the Abelian subalgebra given by X_t, $t \in \mathbb{R}$, let $a_t = \exp(X_t)$ denote the exponential of X_t, and also define $A_\mathfrak{q} = \exp(\mathfrak{a}_\mathfrak{q})$.

Let (considered as row vectors)

$$u = (u_1, \ldots, u_p) \in \mathbb{F}^p \quad \text{and} \quad v = (v_q, \ldots, v_1) \in \mathbb{F}^q,$$

and let $w \in \mathrm{Im}\,\mathbb{F}$ (i. e., $w = 0$ for $\mathbb{F} = \mathbb{R}$). Define $N_{u,v,w} \in \mathfrak{g}$ as the matrix given by

$$N_{u,v,w} = \begin{pmatrix} -w & u & v & w \\ -\overline{u}^T & 0 & 0 & \overline{u}^T \\ \overline{v}^T & 0 & 0 & -\overline{v}^T \\ -w & u & v & w \end{pmatrix}.$$

Then

$$\exp(N_{u,v,w}) = I + N_{u,v,w} + \frac{1}{2} \cdot N_{u,v,w}^2 \,,$$

and a small calculation yields that

$$\begin{aligned} &a_t \exp(N_{u,v,w}) \cdot x_0 = \\ &= \Big(\sinh t + \frac{1}{2} \cdot e^t \,(|u|^2 - |v|^2) + e^t\, w,\ \overline{u}; \\ &-\overline{v},\ \cosh t + \frac{1}{2} \cdot e^t \,(|u|^2 - |v|^2) + e^t\, w\Big)^T , \end{aligned} \tag{1}$$

for any $t \in \mathbb{R}$.

Define the nilpotent subalgebra $\mathfrak{n}^*$ as follows, for $p \geqslant q$,

$$\mathfrak{n}^* = \{N_{u,v,w} : u = (-\overline{v^r}, u'),\ v \in \mathbb{F}^q,\ u' \in \mathbb{F}^{p-q}\}, \tag{2}$$

and, for $p < q$,

$$\mathfrak{n}^* = \{N_{u,v,w} : v = (-\overline{u^r}, v'),\ u \in \mathbb{F}^p,\ v' \in \mathbb{F}^{q-p}\}, \tag{3}$$

where u^r, v^r means that the order of the indices is reversed. By abuse of notation, we leave out the superscript ^{r} in what follows.

We finally also define the following ρ-factors. Let $d = \dim_{\mathbb{R}} \mathbb{F}$, and let

$$\rho_{\mathfrak{q}} = (1/2)(dp + dq + 2(d-1)) \in \mathbb{R}, \quad \rho_1 = (1/2)(|dp - dq| + 2(d-1)) \in \mathbb{R}.$$

Let $N^* = \exp(\mathfrak{n}^*)$ denote the nilpotent subgroup generated by $\mathfrak{n}^*$. For functions f on G/H, we define, assuming convergence,

$$Rf(g) = \int_{N^*} f(gn^*H)\, dn^* \qquad (g \in G). \tag{4}$$

Let $f \in \mathcal{C}^2(G/H)$, the space of L^2-Schwartz functions on G/H. From [1] and [2], we know that the Radon transform Rf is a smooth function. Also, the integral defining R converges uniformly on compact sets, and R is G- and $\mathfrak{g}$-equivariant.

We define the associated Abel transform $\mathcal{A}$ by $\mathcal{A}f(a) = a^{\rho_1} Rf(a)$, for $a \in A_{\mathfrak{q}}$. We are mainly interested in the values of Rf and $\mathcal{A}f$ on the elements a_s, and thus define $Rf(s) = Rf(a_s)$, and, similarly, $\mathcal{A}f(s) = \mathcal{A}f(a_s)$, for $s \in \mathbb{R}$. Let Δ denote the Laplace–Beltrami operator on G/H. Then, for $f \in \mathcal{C}^2(G/H)$,

$$\mathcal{A}(\Delta f) = \left(\frac{d^2}{ds^2} - \rho_{\mathfrak{q}}^2\right) \mathcal{A}f \qquad (s \in \mathbb{R}). \tag{5}$$

Finally, for $R > 0$, let $C_R^\infty(G/H)$ denote the subspace of smooth functions on G/H with support in the (K-invariant) 'ball' $\{ka_s \cdot x_0 \mid |s| \leqslant R\}$ of radius R. Similarly, let $C_R^\infty(\mathbb{R})$ denote the subspace of smooth functions on $\mathbb{R}$ with support in $[-R, R]$, and let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space on $\mathbb{R}$.

§ 3. The discrete series and the Abel transform

Let $q > 1$, or $d > 1$. The discrete series for the projective hyperbolic spaces can then be parametrized as

$$\{T_\lambda \,|\, \lambda = \frac{1}{2}(dq - dp) - 1 + \mu_\lambda > 0,\ \mu_\lambda \in 2\mathbb{Z}\},$$

see [1] and [2]. The spherical discrete series are given by the parameters λ for which $\mu_\lambda \leqslant 0$, including the 'exceptional' discrete series corresponding to $\lambda > 0$ for which $\mu_\lambda < 0$.

For $q = d = 1$, the discrete series is parameterized by $\lambda \in \mathbb{R}\backslash\{0\}$ such that $|\lambda| + \rho_{\mathfrak{q}} \in 2\mathbb{Z}$, and there are no spherical discrete series.

The parameters λ are, via the formula $\Delta f = (\lambda^2 - \rho_{\mathfrak{q}}^2) f$, related to the eigenvalues of Δ acting on functions f in the corresponding representation space in $L^2(G/H)$.

Let D be the G-invariant differential operator $\Delta_\rho(\Delta_\rho - \lambda_1^2) \dots (\Delta_\rho - \lambda_r^2)$, where $\lambda_1, \dots, \lambda_r$ are the parameters of the non-cuspidal discrete series, and $\Delta_\rho = \Delta + \rho_{\mathfrak{q}}^2$.

We have a complete classification of the cuspidal and non-cuspidal discrete series for the projective hyperbolic spaces, also including information about the asymptotics of the Radon and Abel transforms:

Theorem 1. *Let G/H be a projective hyperbolic space over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, with $p \geqslant 0$, $q \geqslant 1$.*

(i) *If $d(q - p) \leqslant 2$, then all discrete series are cuspidal.*

(ii) *If $d(q - p) > 2$, then non-cuspidal discrete series exists, given by the parameters $\lambda > 0$ with $\mu_\lambda \leq 0$. More precisely, if $0 \neq f \in \mathcal{C}^2(G/H)$ belongs to T_λ, then $\mathcal{A}f(s) = Ce^{\lambda s}$, with $C \neq 0$.*

(iii) *T_λ is non-cuspidal if and only if T_λ is spherical.*

(iv) *If $p \geqslant q$, and $f \in C_R^\infty(G/H)$, for $R > 0$, then $\mathcal{A}f \in C_R^\infty(\mathbb{R})$.*

(v) *If $d(q - p) \leqslant 1$, and $f \in \mathcal{C}^2(G/H)$, then $\mathcal{A}f \in \mathcal{S}(\mathbb{R})$.*

(vi) *Assume $d(q - p) > 1$. Then $\mathcal{A}(Df) \in \mathcal{S}(\mathbb{R})$, for $f \in \mathcal{C}^2(G/H)$.*

The above theorem is almost identical to [2, Theorem 6.1], except for item (vi), which was only proved for functions in the (dense) G-invariant subspace $\mathcal{V}$ of $\mathcal{C}^2(G/H)$ generated by the K-irreducible $(K \cap H)$-invariant functions. Additionally, [2, Theorem 6.1] furthermore included the exceptional case corresponding to the Cayley numbers $\mathbb{O}$.

Theorem 1 (including the reformulation of (vi)) also holds for the real non-projective spaces $\mathrm{SO}\,(p + 1, q + 1)_e / \mathrm{SO}\,(p + 1, q)_e$, except for item (iii), due to the existence of non-cuspidal non-spherical discrete series corresponding to negative and odd values of μ_λ in the exceptional series, see [1, Section 5].

The conditions in (vi) essentially state that $\mathcal{A}f$ is a Schwartz function if f is perpendicular to all non-cuspidal discrete series. The factor Δ_ρ, however, seems to be necessary (except in the real case with $q-p$ odd), even for the case $d(q-p)=2$, where there are no non-cuspidal discrete series.

In the next section, we prove Theorem 1(vi).

§ 4. Proof of Theorem 1(vi)

First we note, following [2, Section 10], that the Schwartz decay conditions are satisfied near $-\infty$ for $\mathcal{A}(f)$, and thus also for $\mathcal{A}(Df)$. This leaves us to study the Abel transform near $+\infty$.

Let $f \in \mathcal{C}^2(G/H)$, and write $f[x]=f(gH)$, where $x = g\cdot x_0$. From (1) and (3), we get

$$\begin{aligned} Rf(s) &= \int_{N^*} f(a_s n^* H)dn^* \\ &= \int_{\mathbb{R}^{dq-dp}\times\mathbb{R}^{dp}\times\mathbb{R}^{d-1}} f\left[(\sinh s - 1/2e^s|v'|^2 + e^s w, u;\right. \\ &\qquad \left. -u, -v', \cosh s - 1/2e^s|v'|^2 + e^s w)\right]\, dv'\, du\, dw. \end{aligned}$$

Let $v' = |v'|\overline{v}$, $v = -\sinh s + 1/2e^s|v'|^2$, such that $|v'|^2 = 1 + 2e^{-s}v - e^{-2s}$, and $\overline{w} = e^s w$. Then,

$$\begin{aligned} Rf(s) = e^{-ds} \int_{-\sinh s}^{\infty} d\overline{w} \int_M f\left[(\overline{w} - v, u; -u, -(1+2e^{-s}v - e^{-2s})^{1/2}\overline{v}, e^{-s} - v + \overline{w})\right] \times \\ \times (1 + 2e^{-s}v - e^{-2s})^{(dq-dp)/2-1}\, dv\, d\overline{v}\, du\,, \end{aligned}$$

where $M = \mathbb{S}^{dq-dp-1}\times\mathbb{R}^{dp}\times\mathbb{R}^{d-1}$ and $\mathbb{S}^r$ is the unit sphere in $\mathbb{R}^r$.

We will use the identification of $\mathbb{X} = \mathbb{X}(p+1, q+1; \mathbb{F})$ with

$$\mathbb{X} = \{z \in \mathbb{F}^{p+q+2} : [z,z] < 0\}/\sim,$$

and identify a function f on $\mathbb{X}$ with a homogeneous function of z of degree zero on $\{z\in\mathbb{F}^{p+q+2} : [z,z]<0\}$.

We now identify $\mathbb{F}^{p+q+2}$ with $\mathbb{R}^{d(p+q+2)}$ such that the coordinates satisfy $\operatorname{Re} z_j = x_{dj}$, for $j = 1, \ldots, p+q+2$. Consider the real hyperbolic space

$$\widetilde{\mathbb{X}} = \{z \in \mathbb{F}^{p+q+2} : [z,z] = -1\}.$$

The group $\widetilde{G} = \mathrm{O}(d(p+1), d(q+1))$ acts transitively on $\widetilde{\mathbb{X}}$. Let $\tilde{K}$ denote the standard maximal compact subgroup $\mathrm{O}(d(p+1))\times\mathrm{O}(d(q+1))$ of $\widetilde{G}$. Let $U(\widetilde{\mathfrak{k}})$, respectively $U(\mathfrak{k})$, denote the universal enveloping algebra of the Lie algebra $\widetilde{\mathfrak{k}}$ of $\widetilde{K}$, respectively of the Lie algebra $\mathfrak{k}$ of K.

Lemma 1. *Let* $U \in U(\tilde{\mathfrak{k}})$, *then* U *maps* $\mathcal{C}^2(G/H)$ *into itself.*

Proof. The lemma is obvious for $d = 1$. So assume $d > 1$. We note that any element $x \in \widetilde{\mathbb{X}}$ can be written as $x = ka \cdot x_0$, where $k \in K$, and $a = a_s$, $s \geq 0$. Let $\widetilde{H} = \mathrm{O}(d(p+1), d(q+1)-1)$, and let $\tilde{\mathfrak{m}}$ denote the commutator of $A_{\mathfrak{q}}$ in the Lie algebra of $\widetilde{K} \cap \widetilde{H}$. Then $\tilde{\mathfrak{k}} = \mathfrak{k} + \tilde{\mathfrak{m}}$.

Let $U_k = \mathrm{Ad}\,(k)U$, for $k \in K$, then $Uf = (\mathrm{Ad}\,(k^{-1})U_k)f$. By the Campbell–Baker–Hausdorff formula, there exists an element $U_k^0 \in U(\mathfrak{k})$, such that $U_k = U_k^0$ modulo the left ideal generated by $\tilde{\mathfrak{m}}$. This implies that

$$Uf[ka \cdot x_0] = (\mathrm{Ad}\,(k^{-1})U_k^0)f[ka \cdot x_0].$$

The map $k \mapsto \mathrm{Ad}\,(k^{-1})U_k^0$ is continuous into a finite dimensional subspace of $U(\mathfrak{k})$, and we can write $Uf[ka \cdot x_0] = (\mathrm{Ad}\,(k^{-1})U_k^0)\, f[ka \cdot x_0] = \Sigma_i u_i(k) U_i f[ka \cdot x_0]$, for a finite set of elements $U_i \in U(\mathfrak{k})$ and continuous coefficients $u_i(k)$. It follows that Uf is in $\mathcal{C}^2(G/H)$. ☐

Define for $t = (t_1, t_2, t_3) \in \mathbb{R}^3$, the auxiliary function

$$G_f(t_1, t_2, t_3) = \int_M f\left[(\overline{w} + t_1, u; -u, t_2\overline{v}, t_3 + \overline{w})\right] d\overline{v}\, du\, d\overline{w},$$

and, with the identification $z = e^{-s}$, define the function $F(z) = e^{ds} Rf(s)$. Then, since $\sinh s = -(z - z^{-1})/2$, we get

$$F(z) = \int_{(z-z^{-1})/2}^{\infty} G_f\left(-v, -(1 + 2zv - z^2)^{1/2}, z - v\right) (1 + 2zv - z^2)^{(dq-dp)/2-1}\, dv. \tag{6}$$

Lemma 2. *The function* G_f *is homogeneous of degree* $dp + d - 1$ *on the cone* $t_1^2 - t_2^2 - t_3^2 < 0$, *it is even in* t_2, *and satisfies* $G_f(-t_1, t_2, -t_3) = G_f(t_1, t_2, t_3)$.

Let X *be the differential operator on* $\mathbb{R}^3$ *given by* $t_3\partial/\partial t_2 - t_2\partial/\partial t_3$. *For all* $f \in \mathcal{C}^2(G/H)$, *and all* $k, N \in \mathbb{N}$, *there exists a constant* C, *such that*

$$|X^k G_f(t)| \leq C(t_2^2 + t_3^2)^{-d(q-p)/4}(1 + \log(t_2^2 + t_3^2))^{-N},$$

on the hyperboloid $t_1^2 - t_2^2 - t_3^2 = -1$.

Proof. The first statement follows from the homogeneity of f and the definition of G_f.

As before we identify $\mathbb{F}^{p+q+2}$ with $\mathbb{R}^{d(p+q+2)}$. For $i = d(1+2p)+1, \ldots, d(1+p+q)$, we define the differential operator

$$D_i f[x] = x_{d(p+q+2)} \frac{\partial}{\partial x_i} f[x] - x_i \frac{\partial}{\partial x_{d(p+q+2)}} f[x].$$

This operator is defined by the left action of an element T_i in $\mathrm{O}(d(q+1))$ (with value 1 in the last entry of the i'th row, value -1 in the last entry of the i'th column, and 0 otherwise), and Lemma 1 thus gives that D_i maps $\mathcal{C}^2(G/H)$ into itself.

Let now $\overline{v} = (v_{d(1+2p)+1}, \ldots, v_{d(1+p+q)}) \in \mathbb{S}^{d(q-p)-1}$. The operator

$$Y_{\overline{v}} = \sum_{i=2+2p}^{1+p+q} v_i D_i,$$

also maps $\mathcal{C}^2(G/H)$ into itself, and

$$\left|Y_{\overline{v}} f[x]\right| \leqslant d(q-p) \max_i(\left|D_i f[x]\right|).$$

Applying the operator X to the integrand in the definition of G_f, we get

$$\begin{aligned} Xf[t_1, u; -u, t_2\overline{v}, t_3] &= t_3 \sum \frac{\partial}{\partial x_i} f[.]v_i - t_2 \frac{\partial}{\partial x_{d(p+q+2)}} f[.] \\ &= t_3 \sum \frac{\partial}{\partial x_i} f[.]v_i - t_2 \sum v_i^2 \frac{\partial}{\partial x_{d(p+q+2)}} f[.] \\ &= Y_{\overline{v}} f[t_1, u; -u, t_2\overline{v}, t_3] \end{aligned}$$

the summations are taken over $i = d(1+2p)+1, \ldots, d(1+p+q)$. The inequality for $X^k f$ follows from repeated use of this formula and from the asymptotic estimates of functions in $\mathcal{C}^2(G/H)$. □

In particular, it follows that the function $v \mapsto X^k G_f(-v, -1, -v)$ has the same parity as k.

Lemma 3. *Let k_0 be the largest integer such that $k_0 < (dq-dp)/2$, and let $\epsilon = (dq-dp)/2 - k_0$. Define $t = t(z,v) = (-v, -(1+2zv-z^2)^{1/2}, z-v)$. Then*

(i) For $k \leqslant k_0$, the function

$$v \mapsto \frac{\partial^k}{\partial z^k} \Big(G_f(t(z,v)) \big|(1+2zv-z^2)\big|^{(dq-dp)/2-1} \Big)$$

is uniformly integrable over $\mathbb{R}$ for $z < 1$.

(ii) For $k \leqslant k_0$ odd, this function is an odd function of v for $z = 0$.

Proof. Notice that $t_1^2 - t_2^2 - t_3^2 = -1$ and $t_2^2 + t_3^2 = 1 + v^2$, for $t = t(z,v)$, and that the integral (6) is uniformly convergent for $0 \leqslant z \leqslant k < \infty$. The same holds with G_f replaced by $X^k G_f$.

Repeated use of the formula

$$\begin{aligned} \frac{\partial}{\partial z} G_f(t(z,v))(1+2zv-z^2)^{\alpha} = &- XG_f(t(z,v))(1+2zv-z^2)^{\alpha-1/2} \\ &+ 2\alpha G_f(t(z,v))(1+2zv-z^2)^{\alpha-1}(z-v) \end{aligned}$$

yields (i), and together with the parity properties of $X^k G_f$ also gives (ii). □

We notice that $\epsilon = 1$ if $d(q-p)$ is even, and $\epsilon = 1/2$ if $d(q-p)$ is odd, i. e., if $d = 1$ and $q-p$ is odd.

For $k < k_0$, the derivatives $\partial^k/\partial z^k$ of $G_f(t(z,v))(1+2zv-z^2)^{(dq-dp)/2-1}$ are zero at $v = -\sinh s = (z - z^{-1})/2$, whence the integrand is at least k_0 times differentiable near $z = 0$, and we can compute the derivatives $d^k/dz^k F(z)$ by differentiating under the integral sign in (6).

If $k_0 > 0$, we can use Taylors formula to express $F(z)$ as a polynomial of degree $k_0 - 1$, plus a remainder term involving $d^{k_0}/dz^{k_0} F(\xi)$, for some $0 < \xi(z) < z$,

$$F(z) = c_0 + c_1 z + c_2 z^2 + ... + c_{k_0-1} z^{k_0-1} + R_{k_0}(\xi) z^{k_0},$$

where $0 < \xi < z$, and

$$c_j = \frac{1}{j!} \int_{-\infty}^{\infty} \frac{d^j}{dz^j}\bigg|_{z=0} (G_f(t(z,v))(1+2zv-z^2)^{(dq-dp)/2-1})\, dv,$$

for $j \in \{0, \dots, k_0 - 1\}$. The remainder term is given by:

$$R_{k_0}(\xi) = \frac{1}{k_0!} \int_{(\xi-\xi^{-1})/2}^{\infty} \frac{d^{k_0}}{dz^{k_0}}\bigg|_{z=\xi} (G_f(t(z,v))(1+2zv-z^2)^{(dq-dp)/2-1})\, dv.$$

Consider $\mathcal{A}f(s) = e^{\rho_1 s} Rf(s) = z^{-(\rho_1-d)} F(z)$, which is equal to

$$c_0 z^{-(\rho_1-d)} + c_1 z^{-(\rho_1-d-1)} + c_2 z^{-(\rho_1-d-2)} + ... + c_{k_0-1} z^{-\epsilon} + z^{(-\epsilon+1)} R_{k_0}(\xi).$$

Here we have used that $\rho_1 - d = d(q-p)/2 - 1$. For j even, the exponents $-d(q-p)/2-1-j$, for $j \in \{0, \dots, k_0-1\}$, correspond to the parameters $\lambda_1, \dots, \lambda_r$ for the non-cuspidal discrete series, and $c_j = 0$ for j odd, since the integrand is an odd function.

For the real non-projective hyperbolic spaces the condition concerning the parity j does not hold, but in that case *all* the exponents $-d(q-p)/2 - 1 - j$, for $j \in \{0, \dots, k_0 - 1\}$, correspond to parameters $\lambda_1, \dots, \lambda_r$ for the non-cuspidal discrete series, see [1, Section 3].

From the definition of the differential operator D and (5), we see that $\mathcal{A}(Df)$ at most has a contribution from the remainder term, and further that $\mathcal{A}(Df)$ does not have a constant term at ∞, due to the term d^2/ds^2. If $\epsilon = 1/2$, the remainder term $e^{-1/2s} R_{k_0}(\xi(s))$ is clearly rapidly decreasing, and we are thus left to consider the case $\epsilon = 1$, in which case $k_0 = d(q-p)/2 - 1$.

Consider the constant term $C_{R_{k_0}} = \lim_{s\to\infty} R_{k_0}(e^{-s})$, which could be non-zero. We want to show that $R_{k_0}(\xi) - C_{R_{k_0}}$ is rapidly decreasing at $+\infty$, where $\xi = \xi(s)$, with $0 < \xi < e^{-s}$. We also include the case $k_0 = 0$, where we put $\xi = e^{-s}$.

Define

$$H(z,v) = \frac{d^{k_0}}{dz^{k_0}} (G_f(t(z,v))(1+2zv-z^2)^{k_0}).$$

Then, for $\xi < z < 1$,

$$R_{k_0}(\xi) - C_{R_{k_0}} = \int\limits_{(\xi-\xi^{-1})/2}^{\infty} (H(\xi,v) - H(0,v))\,dv + \int\limits_{-\infty}^{(\xi-\xi^{-1})/2} H(0,v)\,dv \;=\; I_1(\xi) + I_2(\xi).$$

For $I_1(\xi)$, there exists $\xi_1 = \xi_1(\xi, v) < \xi$, such that

$$H(\xi,v) - H(0,v) = \xi \left.\frac{d}{dz}\right|_{z=\xi_1} H(z,v),$$

and we get:

$$I_1(\xi) < z \int_{-\infty}^{\infty} \left| \left.\frac{d}{dz}\right|_{z=\xi_1} H(z,v) \right| dv.$$

By Lemma 3, the integrand is uniformly integrable for $z < 1$, and we conclude that $I_1(\xi)$ is bounded by Ce^{-s}.

For s large, the function $H(0,v)$ is for every $N \in \mathbb{N}$ bounded by

$$|H(0,v)| \leq C(1+v^2)^{-d(q-p)/4} |v|^{k_0} \log(1+v^2)^{-N},$$

for some positive constant C. Using this, we find that

$$I_2(z) < C \int_{\sinh s}^{\infty} v^{-1} (\log(v))^{-N}\,dv = C(N-1)^{-1}(\log(\sinh s))^{-N+1} \leqslant Cs^{-N+1}.$$

It follows that $R_{k_0}(\xi) - C_{R_{k_0}}$ is rapidly decreasing at $+\infty$, whence $\mathcal{A}(Df)$ is rapidly decreasing at $+\infty$, which finishes the proof of Theorem 1.

References

[1] N. B. Andersen, M. Flensted-Jensen and H. Schlichtkrull, "Cuspidal discrete series for semisimple symmetric spaces", *Journal of Functional Analysis*, **263**:8 (2012), 2384–2408.

[2] N. B. Andersen, M. Flensted-Jensen, "Cuspidal discrete series for projective hyperbolic spaces", *Contemporary Mathematics*. V. 598: *Geometric Analysis and Integral Geometry*, Amer Mathematical Society, Providence, 2013, 59–75.

Список литературы

[1] N. B. Andersen, M. Flensted-Jensen and H. Schlichtkrull, "Cuspidal discrete series for semisimple symmetric spaces", *Journal of Functional Analysis*, **263**:8 (2012), 2384–2408.

[2] N. B. Andersen, M. Flensted-Jensen, "Cuspidal discrete series for projective hyperbolic spaces", *Contemporary Mathematics*. V. 598: *Geometric Analysis and Integral Geometry*, Amer Mathematical Society, Providence, 2013, 59–75.

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