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Dual quaternion representation of points, lines and planes

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Abstract. *Background* The bulk of the work on dual quaternions is devoted to their application to describe helical motion. Little attention is paid to the representation of points, lines, and planes (primitives) using them. *Purpose* It is necessary to consistently present the dual quaternion theory of the representation of primitives and refine the mathematical formalism. *Method* It uses the algebra of dual numbers, quaternions and dual quaternions, as well as elements of the theory of screws and sliding vectors. *Results* Formulas have been obtained and systematized that use exclusively dual quaternionic operations and notation to solve standard problems of three-dimensional geometry. *Conclusions* Dual quaternions can serve as a full-fledged formalism for the algebraic representation of a three-dimensional projective space.

Key words and phrases: dual numbers, quaternions, dual quaternions, projective space

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1. Introduction

The parabolic biquaternions (dual quaternions) discussed in this paper were first considered by Clifford. However, they were systematically studied later by E. Studi [1, 2] and A. P. Kotelnikov [3].

The article [4] analyzes a large number of publications and calculates the frequency of mentions of the term dual quaternion. It can be concluded that less than 100 works mentioning dual quaternions were published throughout the 20th century, but already in the early 2000s the number of works increased dramatically. It is worth noting that the authors apparently did not take into account the theory of screws in the calculation, which is related to dual quaternions, but uses a slightly different notation [5–9].

The huge boost in the number of publications can be explained by the development of computer graphics and robotics. It is in these areas that dual quaternions have found their application [10–15], although they were initially developed in works on mechanics.

1.1. Structure of the paper

The paper consists of an introduction, four parts, results and conclusion.

- The first part summarizes the theoretical information about dual numbers.
- The second part contains the necessary theory concerning quaternions. We only avoid the question of using quaternions for rotation in space.
- The third part describes in detail the theory of dual quaternions. The main focus is on notation, so when calculating scalar and helical products, all calculations are described in great detail (it may even seem to some that they are overly detailed).
- In the fourth part, the dual quaternion algebra is used to represent points, lines, and planes in a three-dimensional projective space. Some formulas are derived.
- The main results of the work are summarized in tables, which are given in the section 6.

1.2. Notations and conventions

The following naming conventions are accepted in this article

- Quaternions are indicated by lowercase Latin letters from the end of the alphabet: p, q, r. The components of the quaternions are indicated by the same letters, but with the indexes p_0, p_1, etc.
- Dual quaternions are indicated by uppercase Latin letters from the end of the alphabet: P, Q, R. The components of the quaternions are indicated by the same letters, but with the indexes P_0, P_1, etc.
- Vectors and pure quaternions are indicated by lowercase bold Latin letters: $\mathbf{q}$, $\mathbf{v}$, etc.
- Pure dual quaternions are indicated by uppercase bold Latin letters: $\mathbf{Q}$, $\mathbf{V}$, etc.
- Individual scalars (real numbers) are denoted by the Greek letters α, β, etc.

To avoid ambiguity in the notation system, we do not use multiple quaternions and dual quaternions designated by the same letter, but distinguished by an index. The only exceptions are dual quaternions of points, lines, and planes, the components of which are denoted by letters other than the letters denoting these dual quaternions.

2. Dual numbers

The dual number z is algebraically defined as follows:

$$z = a + \varepsilon b,\ a, b \in \mathbb{R},\ \varepsilon^2 = 0, \varepsilon \neq 0.$$

The number a is called the real or *main* part, and b — is called the imaginary, *dual* or *moment* part. Numbers with zero real part will be called *pure* dual numbers. The special symbol ε is referred to as the dual or parabolic imaginary unit. It is also known as Clifford's complexity symbol [16, p. 43]

2.1. Algebraic form

Using only the definition of the dual unit $\varepsilon^2 = 0$, basic algebraic operations for dual numbers can be introduced $z_1 = a_1 + \varepsilon b_1$ и $z_2 = a_2 + \varepsilon b_2$.

Addition $z_1 + z_2 = (a_1 + a_2) + \varepsilon(b_1 + b_2)$.
Subtraction $z_1 - z_2 = (a_1 - a_2) + \varepsilon(b_1 - b_2)$.
Multiplication $z_1 \cdot z_2 = (a_1 + \varepsilon b_1) \cdot (a_2 + \varepsilon b_2) = a_1 a_2 + \varepsilon a_1 b_2 + \varepsilon a_2 b_1 + \varepsilon^2 b_1 b_2 = a_1 a_2 + \varepsilon(a_1 b_2 + b_1 a_2)$.
Conjugation $\bar{z} = a - \varepsilon b$.

Using the conjugation operation, the squared modulus of a dual number is given by:

$$|z|^2 = z\bar{z} = (a + \varepsilon b) \cdot (a - \varepsilon b) = a^2.$$

Taking the square root of the real number a^2, yields:

Modulus $|z| = |a|$.

The real part of a dual number can be calculated using the formula:

$$\Re z = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}(a + \varepsilon b + a - \varepsilon b) = a.$$

Multiplicative inverse divided by the number z is found by noting that

$$1 = \frac{z\bar{z}}{z\bar{z}} = z\frac{\bar{z}}{z\bar{z}},$$

hence

$$z^{-1} = \frac{\bar{z}}{z\bar{z}} = \frac{a - \varepsilon b}{a^2}.$$

Division of two dual numbers z_1 and z_2 is defined for all z_2 such that $|z_2| \neq 0$:

$$\frac{z_1}{z_2} = \frac{a_1 + \varepsilon b_1}{a_2 + \varepsilon b_2} = \frac{(a_1 + \varepsilon b_1)(a_2 - \varepsilon b_2)}{(a_2 + \varepsilon b_2)(a_2 - \varepsilon b_2)} = \frac{a_1}{a_2} + \varepsilon\frac{b_1 a_2 - b_2 a_1}{a_2^2}.$$

Several important remarks follow from the obtained formulas.

- First, note that the multiplicative inverse is not defined for all nonzero dual numbers z, since the formula for the inverse is valid only when $a \neq 0$.
- It follows from the previous point that the set of dual numbers is not a field, because the requirement of existence of a multiplicative inverse for every nonzero element is not satisfied. Thus, all numbers of the form $z = \varepsilon b$ are nonzero but have no inverse, since $\Re z = 0$.
- Dual numbers of the form $z = b\varepsilon$ are *nontrivial zero divisors*, since for such numbers the equality holds
$$(b_1\varepsilon) \cdot (b_2\varepsilon) = b_1 b_2 \varepsilon^2 = 0,$$
- From the above properties of the algebraic operations, it is clear that in no case do the imaginary parts of numbers contribute to the real part of the result of these operations.

To complement the remark on zero divisors, consider the following definition. An element of a ring $\alpha \in \mathbb{K}$ is called *left zero divisor*, if there exists an element of the ring $\beta \neq 0_{\mathbb{K}}$ such that $\alpha \cdot \beta = 0_{\mathbb{K}}$. Similarly, if $\beta \cdot \alpha = 0_{\mathbb{K}}$, then α is *right zero divisor*. It is obvious that the zero element $0_{\mathbb{K}}$ is both a left and a right zero divisor, since $0_{\mathbb{K}} \cdot \beta = \beta \cdot 0_{\mathbb{K}} = 0_{\mathbb{K}}$. Therefore, $0_{\mathbb{K}}$ is a *trivial* zero divisor, and zero divisors distinct from $0_{\mathbb{K}}$ are called nontrivial or improper.

2.2. "Trigonometric" and "exponential" forms

A dual number z, such that $|z| \neq 0$, can be written in the form:

$$a + \varepsilon b = a\left(1 + \frac{b}{a}\varepsilon\right) = a(1 + \varphi\varepsilon),$$

where $\varphi = \operatorname{Arg} z = b/a$ are called the *argument* or *parameter* of the dual number z. This representation is a kind of analogue of both the trigonometric form and the exponential form of an ordinary complex number. Hereinafter, this form will be referred to as the exponential form of a dual number

For the conjugate number $\bar{z}$, the exponential form is:

$$\bar{z} = a(1 - \varphi\varepsilon),$$

modulus $|\bar{z}| = |z|$, and argument $\operatorname{Arg}\bar{z} = -b/a$.

The analogy with the exponential form of a complex number continues for multiplication and division. For the product of two dual numbers z_1 and z_2:

$$z = a_1(1 + \varphi_1\varepsilon) \cdot a_2(1 + \varphi_2\varepsilon) = a_1a_2\big(1 + (\varphi_1 + \varphi_2)\varepsilon\big),$$

that is, when multiplying, the arguments are added and the modules are multiplied. For division

$$\frac{z_1}{z_2} = \frac{a_1(1 + \varphi_1\varepsilon_1)}{a_2(1 + \varphi_2\varepsilon)} = \frac{a_1(1 + \varphi_1\varepsilon)a_2(1 - \varphi_2\varepsilon)}{a_2(1 + \varphi_2\varepsilon)a_2(1 - \varphi_2\varepsilon)} = \frac{a_1\big(1 + (\varphi_1 - \varphi_2)\varepsilon\big)}{a_2\big(1 - \varphi_2\varepsilon + \varphi_2\varepsilon\big)} = \frac{a_1}{a_2}(1 + (\varphi_1 - \varphi_2)\varepsilon),$$

that is, when dividing, the arguments are subtracted and the modules are divided. In the case of division, the number z_2 must have nonzero modulus $|z_2| \neq 0$.

In exponential form, raising to a natural power n has a particularly simple expression:

$$z^n = \big(a(1 + \varphi\varepsilon)\big)^n = a \cdot a \cdot \ldots \cdot a\big(1 + (\varphi + \varphi + \ldots + \varphi)\big) = a^n(1 + n\varphi\varepsilon) = a^n + \varepsilon n a^{n-1} b.$$

To find $\sqrt[n]{z} = \sqrt[n]{a(1 + \varphi\varepsilon)}$ assume that $\sqrt[n]{z} = z_1$, Then, by definition $z_1^n = z$, hence

$$a_1^n(1 + n\varphi_1\varepsilon) = z = a(1 + \varphi\varepsilon),$$

which yields $a_1 = \sqrt[n]{a}$ and $\varphi_1 = \varphi/n$, and therefore

$$\sqrt[n]{z} = \sqrt[n]{r}\Big(1 + \frac{\varphi}{n}\varepsilon\Big).$$

Note that for odd n the root $\sqrt[n]{z}$ always exists, whereas for even n it exists only for z with nonnegative modulus $|z| = a \neq 0$. In algebraic form, the formula for the root is:

$$\sqrt[n]{a + b\varepsilon} = \sqrt[n]{a} + \frac{ba^{\frac{1-n}{n}}}{n}\varepsilon, \quad \text{в частности} \quad \sqrt{a + b\varepsilon} = \sqrt{a} + \frac{b}{2\sqrt{a}} = \sqrt{a}\Big(1 + \frac{b}{2a}\Big). \tag{1}$$

It is noteworthy that neither raising to a power nor taking a root leads to any contribution of the dual parts to the real parts of the results.

2.3. Matrix form

All algebraic operations on dual numbers can be reduced to matrix operations by setting

$$\varepsilon \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad z = a + b\varepsilon \leftrightarrow \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Then, for example

$$z_1 \cdot z_2 \leftrightarrow \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + a_2b_1 \\ 0 & a_1a_2 \end{pmatrix} \leftrightarrow a_1a_2 + (a_1b_2 + a_2b_1)\varepsilon.$$

Theoretically, from a computational point of view, such a reduction can be justified when the programming language supports vectorized matrix operations. In practice, however, the performance gain is usually not significant.

Table 1
Trigonometric and inverse trigonometric functions of dual numbers expressed through functions of a real variable.

$\sin(a+\varepsilon b) = \sin a + b\varepsilon \cos a$	$\arcsin(a+\varepsilon b) = \arcsin a + \dfrac{b\varepsilon}{\sqrt{1-a^2}}$
$\cos(a+\varepsilon b) = \cos a - b\varepsilon \sin a$	$\arccos(a+\varepsilon b) = \arccos a - \dfrac{b\varepsilon}{\sqrt{1-a^2}}$
$\operatorname{tg}(a+\varepsilon b) = \operatorname{tg} a + \dfrac{b\varepsilon}{\cos^2 a}$	$\operatorname{arctg}(a+\varepsilon b) = \operatorname{arctg} a + \dfrac{b\varepsilon}{1+a^2}$
$\operatorname{ctg}(a+\varepsilon b) = \operatorname{ctg} a - \dfrac{b\varepsilon}{\sin^2 a}$	$\operatorname{arcctg}(a+\varepsilon b) = \operatorname{arctg} a - \dfrac{b\varepsilon}{1+a^2}$

2.4. Taylor expansion

Using the defining property of the dual imaginary unit $\varepsilon^2 = \varepsilon^3 = \ldots = \varepsilon^n = 0$ for any natural power, consider the Maclaurin series for the exponential of a pure dual number:

$$\exp(b\varepsilon) = \sum_{n=0}^{\infty} \frac{(b\varepsilon)^n}{n!} = 1 + b\varepsilon + \frac{b^2\varepsilon^2}{2!} + \ldots = 1 + b\varepsilon,$$
$$\exp(a + b\varepsilon) = e^a e^{b\varepsilon} = e^a(1 + b\varepsilon).$$

A more general formula is derived from the formal Taylor series for the function $f(z)$ at the point a:

$$f(a+\varepsilon b) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(a+\varepsilon b - a)^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\varepsilon^n b^n}{n!} =$$
$$= f(a) + f'(a)b\varepsilon + \frac{f''(a)\varepsilon^2 b^2}{2!} + \ldots = f(a) + f'(a)b\varepsilon.$$

This yields an extremely important formula:

$$f(a+\varepsilon b) = f(a) + f'(a)b\varepsilon,$$

which provides a method for calculating the values of functions from a dual number, if the value of the derivative of the real part of the number $f'(a)$ is known. On the other hand, the same formula allows to calculate the value of the derivative for the value of the argument equal to a, which is used in automatic differentiation algorithms.

2.5. Elementary Functions of Dual Numbers

The formula $f(a+\varepsilon b) = f(a) + f'(a)b\varepsilon$ makes it possible to extend elementary functions to the set of dual numbers, since the right-hand side of the formula contains only the values of the function f from the real number a. For illustration, a brief summary of some basic elementary functions is presented in tables 1 and 2.

3. Quaternion algebra

The theory of quaternions is well known. We will highlight the following books [17–19]. Next, we briefly outline the quaternion algebra in order to coordinate the notation and basic concepts with the further presentation of the biquaternion algebra.

Table 2

Power functions, exponent, and logarithm of dual numbers

$(a+\varepsilon b)^n = a^n + na^{n-1}b\varepsilon$	$\exp(a+\varepsilon b) = e^a + \varepsilon b e^a$
$\sqrt[n]{a+b\varepsilon} = \sqrt[n]{a}\left(1+\dfrac{b\varepsilon}{na}\right)$	$\log_c(a+\varepsilon b) = \log_c a + \dfrac{b\varepsilon}{a\ln a}$

3.1. Basic concepts of quaternion algebra

3.1.1. Basis elements of a quaternion

Let us denote the neutral element of the quaternion algebra by o (lowercase O). Geometrically, it is associated with the origin (the center of the coordinate system) in three-dimensional space $\mathbb{R}^3$. Algebraically, it also serves as the scalar unit [19, 20]. Using this notation, a quaternion is written in the following form:

$$q = q_0\mathrm{o} + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k},$$

Where q_0, q_1, q_2, q_3 are some real numbers. The quaternion q can also be associated with a point in projective space, written in homogeneous coordinates $(q_1, q_2, q_3 \mid q_0)$ [20].

In turn, the basis element o is associated with the proper (finite) point of the origin of coordinates, and the basis elements i, j, k with points at infinity:

$$\mathrm{o} \leftrightarrow O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathrm{i} \leftrightarrow \vec{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathrm{j} \leftrightarrow \vec{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathrm{k} \leftrightarrow \vec{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

A quaternion of the form $p = \mathrm{o} + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$ is associated with an affine point having homogeneous coordinates $(x, y, z \mid 1)$ and Cartesian coordinates (x, y, z).

A pure quaternion of the form $v = v_x\mathrm{i} + v_y\mathrm{j} + v_z\mathrm{k}$ is associated with a point at infinity with coordinates $(v_x, v_y, v_z \mid 0)$ or with a free vector $\mathbf{v} = (v_x, v_y, v_z)^T$ in Cartesian space $\mathbb{R}^3$.

3.1.2. Quaternion multiplication

To derive the formula for quaternion multiplication, it is sufficient to derive the multiplication table for the basis elements $\langle \mathrm{o}, \mathrm{i}, \mathrm{j}, \mathrm{k} \rangle$. In turn, for this it is sufficient to the axiomatic relation introduced by Hamilton:

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1\mathrm{o} = -\mathrm{o}.$$

Let us add the equality $\mathrm{o}^2 = \mathrm{o}$ to it, since algebraically o is a scalar unit. Then the multiplication table of the quaternion basis elements takes the form (2)

	o	i	j	k
o	o	i	j	k
i	i	−o	k	−j
j	j	−k	−o	i
k	k	j	−i	−o

(2)

Now the multiplication of quaternions can be performed using table 2 by expanding the brackets for the replacement of the products of the basis elements:

$$\begin{gathered}
pq = (p_0\mathrm{o} + p_1\mathrm{i} + p_2\mathrm{j} + p_3\mathrm{k})(q_0\mathrm{o} + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}) = \\
+ p_0q_0\mathrm{o} + p_0q_1\mathrm{i} + p_0q_2\mathrm{j} + p_0q_3\mathrm{k} + p_1q_0\mathrm{io} + p_1q_1\mathrm{ii} + p_1q_2\mathrm{ij} + \\
+ p_1q_3\mathrm{ik} + p_2q_0\mathrm{jo} + p_2q_1\mathrm{ji} + p_2q_2\mathrm{jj} + p_2q_3\mathrm{jk} + p_3q_0\mathrm{ko} + p_3q_1\mathrm{ki} + p_3q_2\mathrm{kj} + p_3q_3\mathrm{kk} = \\
= p_0q_0\mathrm{o} + p_0q_1\mathrm{i} + p_0q_2\mathrm{j} + p_0q_3\mathrm{k} + p_1q_0\mathrm{i} - p_1q_1\mathrm{o} + p_1q_2\mathrm{k} - p_1q_3\mathrm{j} + p_2q_0\mathrm{j} - p_2q_1\mathrm{k} - p_2q_2\mathrm{o} + p_2q_3\mathrm{i} + \\
+ p_3q_0\mathrm{k} + p_3q_1\mathrm{j} - p_3q_2\mathrm{i} - p_3q_3\mathrm{o}.
\end{gathered}$$

Next, we will bring similar and group the terms around the basis elements:

$$\begin{gathered}
pq = (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3)\mathrm{o} + (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)\mathrm{i} + \\
+ (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1)\mathrm{j} + (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)\mathrm{k} = \\
= \big(p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3)\big)\mathrm{o} + p_0(q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}) + \\
+ q_0(p_1\mathrm{i} + p_2\mathrm{j} + p_3\mathrm{k}) + (p_2q_3 - p_3q_2)\mathrm{i} + (p_3q_1 - p_1q_3)\mathrm{j} + (p_1q_2 - p_2q_1)\mathrm{k}.
\end{gathered}$$

A more concise form can be written using the scalar and vector products:

$$pq = (p_0q_0 - (\mathbf{p}, \mathbf{q}))\mathrm{o} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}.$$

Particularly, for pure quaternions $p = 0\mathrm{o} + \mathbf{p}$ and $q = 0\mathrm{o} + \mathbf{q}$, the formula simplifies to:

$$\mathbf{pq} = -(\mathbf{p}, \mathbf{q})\mathrm{o} + \mathbf{p} \times \mathbf{q}.$$

Using quaternion multiplication, let us compute the square of a quaternion:

$$p^2 = pp = (p_0^2 - \|\mathbf{p}\|^2) + p_0\mathbf{p} + p_0\mathbf{p} + \mathbf{p} \times \mathbf{p} = (p_0^2 - \|\mathbf{p}\|^2)\mathrm{o} + 2p_0\mathbf{p}. \tag{3}$$

The square of a pure quaternion coincides with the square of its norm taken with a minus sign and is a scalar number:

$$\mathbf{p}^2 = \mathbf{pp} = -\|\mathbf{p}\|^2\mathrm{o}$$

For pure quaternions, the scalar and vector products can be expressed through quaternion multiplication. Consider two pure quaternions $\mathbf{p}$ and $\mathbf{q}$ and write

$$\begin{aligned}
\mathbf{pq} &= -(\mathbf{p}, \mathbf{q})\mathrm{o} + \mathbf{p} \times \mathbf{q}, \\
\mathbf{qp} &= -(\mathbf{q}, \mathbf{p})\mathrm{o} + \mathbf{q} \times \mathbf{p} = -(\mathbf{q}, \mathbf{p})\mathrm{o} - \mathbf{q} \times \mathbf{p}.
\end{aligned}$$

From which stems:

$$\mathbf{pq} + \mathbf{qp} = -2(\mathbf{p}, \mathbf{q})\mathrm{o} \quad \text{and} \quad \mathbf{pq} - \mathbf{qp} = \mathbf{p} \times \mathbf{q} + \mathbf{p} \times \mathbf{q},$$

$$-(\mathbf{p}, \mathbf{q})\mathrm{o} = \frac{1}{2}(\mathbf{pq} + \mathbf{qp}) \quad \text{and} \quad \mathbf{p} \times \mathbf{q} = \frac{1}{2}(\mathbf{pq} - \mathbf{qp}).$$

The expression for the vector product remains valid for quaternions of general form:

$$\begin{matrix}
pq = (p_0q_0 - (\mathbf{p}, \mathbf{q}))\mathrm{o} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q} \\
qp = (p_0q_0 - (\mathbf{p}, \mathbf{q}))\mathrm{o} + q_0\mathbf{p} + p_0\mathbf{q} + \mathbf{q} \times \mathbf{p}
\end{matrix} \Rightarrow \mathbf{p} \times \mathbf{q} - \mathbf{q} \times \mathbf{p} = 2\mathbf{p} \times \mathbf{q} = pq - qp,$$

$$\mathbf{p} \times \mathbf{q} = \frac{1}{2}(pq - qp).$$

Let us compute the triple quaternion product of pure quaternions **qpq**, which we will also call the sandwich formula without conjugation.

$$\mathbf{qpq} = \mathbf{q}\big(- (\mathbf{p},\mathbf{q})\mathrm{o} + \mathbf{p}\times\mathbf{q}\big) =$$
$$= -(\mathbf{p},\mathbf{q})\mathbf{q} + \mathbf{q}(\mathbf{p}\times\mathbf{q}) = -(\mathbf{p},\mathbf{q})\mathbf{q} - (\mathbf{q},\mathbf{p}\times\mathbf{q})\mathrm{o} + \mathbf{q}\times(\mathbf{p}\times\mathbf{q}) =$$
$$= -(\mathbf{p},\mathbf{q})\mathbf{q} + (\mathbf{q},\mathbf{q})\mathbf{p} - (\mathbf{p},\mathbf{q})\mathbf{q} = -2(\mathbf{p},\mathbf{q})\mathbf{q} + (\mathbf{q},\mathbf{q})\mathbf{p} = \|\mathbf{q}\|^2\mathbf{p} - 2(\mathbf{p},\mathbf{q})\mathbf{q}.$$

$$\mathbf{qpq} = \|\mathbf{q}\|^2\mathbf{p} - 2(\mathbf{p},\mathbf{q})\mathbf{q}. \tag{4}$$

Formula (4) will allow simplifying calculations when computing the reflection of a point relative to a plane[1][2].

3.1.3. Quaternion conjugation

Let us introduce the operation of quaternion *conjugation*. If a quaternion $p = p_0\mathrm{o} + \mathbf{p}$ is given, then its conjugate is defined by the following formula:

$$p^* = p_0\mathrm{o} - \mathbf{p} = p_0\mathrm{o} - p_1\mathrm{i} - p_2\mathrm{j} - p_3\mathrm{k}.$$

To follow up, we compute:

$$pp^* = (p_0\mathrm{o} + \mathbf{p})(p_0\mathrm{o} - \mathbf{p}) = p_0^2\mathrm{o} - p_0\mathbf{p} + p_0\mathbf{p} - \mathbf{pp} = p_0^2\mathrm{o} - \mathbf{pp},$$

Where $\mathbf{pp} = -(\mathbf{p},\mathbf{p})\mathrm{o} + \mathbf{p}\times\mathbf{p} = -(\mathbf{p},\mathbf{p})\mathrm{o}$, which allows us to write:

$$pp^* = p_0^2\mathrm{o} + (\mathbf{p},\mathbf{p})\mathrm{o} = \big(p_0^2 + (\mathbf{p},\mathbf{p})\big)\mathrm{o} = (p_0^2 + p_1^2 + p_2^2 + p_3^2)\mathrm{o}.$$

The *module* of a quaternion is the expression

$$|p| = \sqrt{pp^*} = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2},$$

and the *norm* of a pure quaternion is the expression

$$\|\mathbf{p}\| = \sqrt{(\mathbf{p},\mathbf{p})} = \sqrt{p_1^2 + p_2^2 + p_3^2}.$$

Let us show that

$$(pq)^* = q^*p^*,$$

using the formula of quaternion multiplication:

$$(pq)^* = \big[(p_0q_0 - (\mathbf{p},\mathbf{q}))\mathrm{o} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}\big]^* =$$
$$= \big(p_0q_0 - (\mathbf{p},\mathbf{q})\big)\mathrm{o} - p_0\mathbf{q} - q_0\mathbf{p} - \mathbf{p}\times\mathbf{q} =$$
$$= \big(p_0q_0 - (-\mathbf{p},-\mathbf{q})\big)\mathrm{o} + p_0(-\mathbf{q}) + q_0(-\mathbf{p}) - (-\mathbf{p})\times(-\mathbf{q}) =$$
$$= \big(p_0q_0 - (-\mathbf{q},-\mathbf{p})\big)\mathrm{o} + p_0(-\mathbf{q}) + q_0(-\mathbf{p}) + (-\mathbf{q})\times(-\mathbf{p}) = q^*p^*.$$

Using the expression $(pq)^* = q^*p^*$ we prove the property of the module of the product of quaternions:

$$|pq| = \sqrt{(pq)(pq)^*} = \sqrt{pqq^*p^*} = \sqrt{p|q|^2p^*} = \sqrt{|p|^2|q|^2} = |p||q| \Rightarrow |pq| = |p||q|.$$

3.1.4. Scalar product of quaternions

Following [20] we introduce the operation of scalar product of two quaternions of general form. Consider two quaternions $p = p_0 \mathrm{o} + p_1 \mathrm{i} + p_2 \mathrm{j} + p_3 \mathrm{k}$ and $q = q_0 \mathrm{o} + q_1 \mathrm{i} + q_2 \mathrm{j} + q_3 \mathrm{k}$. We compute:

$$pq^* = \big(p_0 q_0 + (\mathbf{p}, \mathbf{q})\big)\mathrm{o} - p_0 \mathbf{q} + q_0 \mathbf{p} - \mathbf{p} \times \mathbf{q},$$

$$qp^* = \big(p_0 q_0 + (\mathbf{p}, \mathbf{q})\big)\mathrm{o} + p_0 \mathbf{q} - q_0 \mathbf{p} - \mathbf{q} \times \mathbf{p} = \big(p_0 q_0 + (\mathbf{p}, \mathbf{q})\big)\mathrm{o} + p_0 \mathbf{q} - q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q}.$$

We calculate the sum of the products

$$pq^* + qp^* = 2\big(p_0 q_0 + (\mathbf{p}, \mathbf{q})\big)\mathrm{o} \Rightarrow \big(p_0 q_0 + (\mathbf{p}, \mathbf{q})\big)\mathrm{o} = \frac{1}{2}\big(pq^* + qp^*\big).$$

Let us define the *scalar product of quaternions* by the following formula

$$(p, q) = \frac{1}{2}(pq^* + qp^*) = \big(p_0 q_0 + (\mathbf{p}, \mathbf{q})\big)\mathrm{o} = p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3.$$

It is easy to see that the definition of the scalar product of quaternions is consistent with the definition of the module of a quaternion:

$$|p|^2 = (p, p) = \big(p_0^2 + \|\mathbf{p}\|^2\big).$$

3.1.5. Unit quaternion

Consider a quaternion $q = q_0 \mathrm{o} + q_1 \mathrm{i} + q_2 \mathrm{j} + q_3 \mathrm{k}$ with a unit module $|q|^2 = q_0^2 + \|q\|^2 = 1$. Always there is such a value θ that $q_0^2 = \cos^2\theta$ and $\|\mathbf{q}\|^2 = \sin^2\theta$ due to the basic trigonometric identity

$$|q|^2 = q_0^2 + |\mathbf{q}|^2 = \cos^2\theta + \sin^2\theta = 1.$$

Let us note that the vector part $\mathbf{q}$ may not be a unit vector, but it can always be expressed through a unit vector $\mathbf{u}$ as follows:

$$\frac{\mathbf{q}}{\|\mathbf{q}\|} = \mathbf{u} \Rightarrow \mathbf{q} = \mathbf{u}\|\mathbf{q}\| = \sin\theta \mathbf{u},$$

Where $\mathbf{u} = u_1 \mathrm{i} + u_2 \mathrm{j} + u_3 \mathrm{k}$ and $\|\mathbf{u}\| = 1$. The unit quaternion can then be written in the trigonometric form: $q = \cos\theta + \sin\theta\mathbf{u}$.

The following terminology is sometimes used. The module of an arbitrary quaternion $|q|$ is called the *tensor* of the quaternion, and the normalized quaternion

$$\frac{q}{|q|} = \frac{q_0}{|q|} + \frac{\mathbf{q}}{|q|},$$

is called the *versor* of the quaternion q. Also, if the quaternion is a unit quaternion to begin with, it can be simply called a *versor*.

Let us denote the unit quaternion as u and write it in the following form:

$$u = \cos\theta + \sin\theta\mathbf{u} = \cos\theta + u_1 \sin\theta \mathrm{i} + u_2 \sin\theta \mathrm{j} + u_3 \sin\theta \mathrm{k}.$$

Any non-unit quaternion can be expressed through its module and versor by dividing both sides of the equation $|q|^2 = q_0^2 + \|q\|^2 = 1$ by $|q|^2$, we get:

$$1 = \frac{|q|^2}{|q|^2} = \frac{q_0^2}{|q|^2} + \frac{\|\mathbf{q}\|^2}{|q|^2} \;\Rightarrow\; q_0^2 = |q|^2\cos^2\theta, \;\; \|\mathbf{q}\|^2 = |q|^2\sin^2\theta.$$

Now we normalize the vector part of the quaternion $\mathbf{q}$ and write:

$$\frac{\mathbf{q}}{\|\mathbf{q}\|} = \mathbf{u} \quad \Rightarrow \mathbf{q} = \|\mathbf{q}\|\mathbf{u} = |q| \sin\theta\mathbf{u}.$$

Therefore, any quaternion can be expressed through its module $|q|$ and versor $\mathbf{u}$ as follows:

$$q = q_0 + \mathbf{q} = |q|\cos\theta + |q|\cos\theta\mathbf{u} = |q|(\cos\theta + \sin\theta\mathbf{u}) = |q|(\cos\theta + u_1\sin\theta\mathrm{i} + u_2\sin\theta\mathrm{j} + u_3\sin\theta\mathrm{k}).$$

Using the formula (3) we compute the square of a unit quaternion:

$$u^2 = uu = (u_0^2 - \sin^2\theta\|\mathbf{u}\|) + 2u_0\sin\theta\mathbf{u} = (\cos^2\theta - \sin^2\theta) + 2\cos\theta\sin\theta\mathbf{u} = \cos 2\theta + \sin 2\theta\mathbf{u}.$$

It turns out that to square a unit quaternion it is enough to double the parameter θ.

Let us compute the product of two unit quaternions $u_1 = \cos\theta_1 + \sin\theta_1\mathbf{u}_1$ and $u_2 = \cos\theta_2 + \sin\theta_2\mathbf{u}_2$

$$u_1u_2 = \big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2(\mathbf{u}_1, \mathbf{u}_2)\big) + \cos\theta_1\sin\theta_2\mathbf{u}_2 + \cos\theta_2\sin\theta_1\mathbf{u}_1 + \mathbf{u}_1\times\mathbf{u}_2\sin\theta_1\sin\theta_2.$$

If the unit quaternions differ only by the parameter θ and have the same unit vector part, that is $\mathbf{u}_1 = \mathbf{u}_2$, then the multiplication formula simplifies significantly:

$$u_1u_2 = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\mathbf{u} = \cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)\mathbf{u}.$$

From this formula it follows that the unit quaternions $u_1 = \cos\theta_1 + \sin\theta_1\mathbf{u}$ and $u_2 = \cos\theta_2 + \sin\theta_2\mathbf{u}$ commute when multiplied. It also allows calculating an arbitrary power of a unit quaternion:

$$u^n = \cos n\theta + \sin n\theta\mathbf{u}.$$

And write an analogue of the formula of Moivre for an arbitrary quaternion with a versor u:

$$q^n = |q|^n(\cos n\theta + \sin n\theta\mathbf{u}).$$

Let us consider a unit pure quaternion $\mathbf{u} = u_1\mathrm{i} + u_2\mathrm{j} + u_3\mathrm{k}$ and compute its square:

$$\mathbf{u}^2 = \mathbf{uu} = -(\mathbf{u}, \mathbf{u}) + \mathbf{u}\times\mathbf{u} = -\|\mathbf{u}\|^2 = -1.$$

We obtained that a unit pure quaternion has a property that defines the elliptic imaginary unit.

Let us prove the following formulas for an arbitrary quaternion q, pure quaternions $\mathbf{u}$, $\mathbf{v}$ and a unit pure quaternion $\mathbf{n}$.

The formula $q(\mathbf{u}\times\mathbf{v})q^{-1} = (q\mathbf{u}q^{-1})\times(q\mathbf{v}q^{-1})$, is valid due to the following chain of equalities:

- $q(\mathbf{u}\times\mathbf{v})q^{-1} = (q\mathbf{u}q^{-1})\times(q\mathbf{v}q^{-1})$,
- $q(\mathbf{u}, \mathbf{v})q^{-1} = (q\mathbf{u}q^{-1}, q\mathbf{v}q^{-1})$,
- $\mathbf{n}(\mathbf{u}\times\mathbf{v})\mathbf{n} = -(\mathbf{nun})\times(\mathbf{nvn})$,
- $(\mathbf{u}\times\mathbf{v})^* = -\mathbf{u}\times\mathbf{v}$.

The formula $q(\mathbf{u}, \mathbf{v})q^{-1} = (q\mathbf{u}q^{-1}, q\mathbf{v}q^{-1})$ is valid, because:

$$q(\mathbf{u}\times\mathbf{v})q^{-1} = \frac{1}{2}q(\mathbf{uv} - \mathbf{vu})q^{-1} = \frac{1}{2}(q\mathbf{uv}q^{-1} - q\mathbf{vu}q^{-1}) =$$
$$= \frac{1}{2}\left(q\mathbf{u}q^{-1}q\mathbf{v}q^{-1} - q\mathbf{v}q^{-1}q\mathbf{u}q^{-1}\right) = (q\mathbf{u}q^{-1})\times(q\mathbf{v}q^{-1}).$$

Formula $q(\mathbf{u}, \mathbf{v})q^{-1} = (q\mathbf{u}q^{-1}, q\mathbf{v}q^{-1})$ is valid because:

$$q(\mathbf{u},\mathbf{v})q^{-1} = -\frac{1}{2}q(\mathbf{uv}+\mathbf{vu})q^{-1} = -\frac{1}{2}(q\mathbf{uv}q^{-1} + q\mathbf{vu}q^{-1}) =$$
$$= -\frac{1}{2}(q\mathbf{u}q^{-1}q\mathbf{v}q^{-1} + q\mathbf{v}q^{-1}q\mathbf{u}q^{-1}) = (q\mathbf{u}q^{-1}, q\mathbf{v}q^{-1}).$$

For a unit pure quaternion $\mathbf{n}$ the equality holds:

$$\mathbf{n}(\mathbf{u}\times\mathbf{v})\mathbf{n} = -(\mathbf{nun})\times(\mathbf{nvn}),$$

To prove this, we will use the fact that $\mathbf{nn} = -1$ and perform a series of transformations:

$$\mathbf{n}(\mathbf{u}\times\mathbf{v})\mathbf{n} = \mathbf{n}\frac{1}{2}(\mathbf{uv}-\mathbf{vu})\mathbf{n} = \frac{1}{2}(\mathbf{nuvn}-\mathbf{nvun}) = \frac{1}{2}(-\mathbf{nunnvn}+\mathbf{nvnnun}) =$$
$$= \frac{1}{2}(\mathbf{nvnnun}-\mathbf{nunnvn}) = (\mathbf{nvn})\times(\mathbf{nun}) = -(\mathbf{nun})\times(\mathbf{nvn}).$$

The formula $(\mathbf{u}\times\mathbf{v})^* = -\mathbf{u}\times\mathbf{v}$ can be derived using the fact that $\mathbf{u}^* = -\mathbf{u}$:

$$(\mathbf{u}\times\mathbf{v})^* = \frac{1}{2}(\mathbf{uv}-\mathbf{vu})^* = \frac{1}{2}\Big((\mathbf{uv})^*-(\mathbf{vu})^*\Big) = \frac{1}{2}(\mathbf{v}^*\mathbf{u}^*-\mathbf{u}^*\mathbf{v}^*) = -\frac{1}{2}(\mathbf{u}^*\mathbf{v}^*-\mathbf{v}^*\mathbf{u}^*) = -\mathbf{u}\times\mathbf{v}.$$

4. Dual quaternions

4.1. Definition of dual dual quaternions

Consider a dual number of the form:

$$Q = q + q^o\varepsilon,$$

where the coefficients q and q^o are quaternions. The quaternion $q = q_0 + \mathbf{q} = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$ is called the *main part*, and the quaternion $q^o = q^o_0 + \mathbf{q}^o = q^o_0 + q^o_1\mathrm{i} + q^o_2\mathrm{j} + q^o_3\mathrm{k}$ is *moment part*. A hypercomplex number Q constructed in this way is called a *parabolic* or *dual* dual quaternion [21, p. 38, 22, p. 124, 16, p. 66, 20].

If both quaternions q and q^o are pure, that is, have zero scalar parts $q_0 = p^o_0 = 0$, then the dual quaternion $\mathbf{Q} = \mathbf{q} + \mathbf{q}^o\varepsilon$ is also called a *pure dual quaternion* and coincides with the notion of a *motor* in screw theory [22, p. 84, 16, p. 169]. The terms *dyad*, bivector [16], dual vector, and line vector [6, p. 5] are also used. The term “bivector” is currently used in the literature to denote an object of a different type.

A dual quaternion Q can be written as a hypercomplex number with eight components:

$$Q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} + q^o_0\varepsilon + q^o_1\mathrm{i}\varepsilon + q^o_2\mathrm{j}\varepsilon + q^o_3\mathrm{k}\varepsilon,$$

with eight basis elements $\langle 1, \mathrm{i}, \mathrm{j}, \mathrm{k}, \varepsilon, \mathrm{i}\varepsilon, \mathrm{j}\varepsilon, \mathrm{k}\varepsilon\rangle$.

To define the dual quaternion product, a multiplication table for the basis elements is required. To do this, in addition to the associativity axiom and the axiomatic relations $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$ and $\varepsilon^2 = 0$, it is necessary to define the commutativity of quaternion imaginary units with the dual unit ε, that is

$$\mathrm{i}\varepsilon = \varepsilon\mathrm{i},\ \mathrm{j}\varepsilon = \varepsilon\mathrm{j},\ \mathrm{k}\varepsilon = \varepsilon\mathrm{k}.$$

The full 8×8 multiplication table for the basis elements of a dual quaternion then has the form shown in table (5).

	1	i	j	k	ε	iε	jε	kε
1	1	i	j	k	ε	iε	jε	kε
i	i	−1	k	−j	iε	−ε	kε	−jε
j	j	−k	−1	i	jε	−kε	−ε	iε
k	k	j	−i	−1	kε	jε	−iε	−ε
ε	ε	iε	jε	kε	0	0	0	0
iε	iε	−ε	kε	−jε	0	0	0	0
jε	jε	−kε	−ε	iε	0	0	0	0
kε	kε	jε	−iε	−ε	0	0	0	0

(5)

Two additional remarks are in order. Firstly, assuming anticommutativity of imaginary units instead of commutativity leads to a contradiction. For example, if ε is presumed to anticommute with i and j, this entails commutativity between ε and k, since

$$\mathrm{i}\varepsilon = -\varepsilon\mathrm{i}, \mathrm{j}\varepsilon = -\varepsilon\mathrm{j} \Rightarrow \varepsilon\mathrm{k} = \varepsilon\mathrm{ij} = -\mathrm{i}\varepsilon\mathrm{j} = \mathrm{ij}\varepsilon = \mathrm{k}\varepsilon.$$

Secondly, it is possible not to introduce the requirement for commutativity between ε and i, j, k, However, in that case, it will be necessary to write the dual unit on the left in all the relations. For example, the dual quaternion Q will have to be defined exclusively as $q + \varepsilon q^o$ and permutations leading to expressions of the type $q^o\varepsilon$ must not be allowed anywhere. This rule is followed, for example, in [16], but it complicates the calculations and does not provide any particular advantage.

Another way to define dual quaternions involves applying the Cayley–Dickson doubling procedure [23–25], generalized to an arbitrary type of complex numbers.

The quaternion $q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$ undergoes transformation wherein the real coefficients q_0, q_1, q_2, q_3 are substituted with dual numbers Q_0, Q_1, Q_2, Q_3 through the application of the doubling procedure.

$$Q = Q_0 + Q_1\mathrm{i} + Q_2\mathrm{j} + Q_3 k = Q_0 + \mathbf{Q}, \quad Q_i = q_i + q_i^o\varepsilon, q_i, q_i^o \in \mathbb{R}, \ i = 0, 1, 2, 3,$$

where Q_0 — *scalar* part (dual number), and $\mathbf{Q}$ — *screw* part. The screw part is also called a dual vector, a *screw* or *pure dual quaternion*. This representation will be called the *dual representation*. In dual representation, all dual quaternion formulas coincide in form with the corresponding quaternion relations, with real numbers replaced by dual numbers.

Terminological remarks. The term biquaternion covers three subtypes [20]:

- if the coefficients of the quaternion are elliptic complex numbers with imaginary unit $\mathrm{i}^2 = -1$, the biquaternion is called an *elliptic biquaternion*;
- In the case of parabolic coefficients (dual numbers with imaginary unit $\varepsilon^2 = 0, \ \varepsilon \neq 0$) the biquaternion is called a parabolic or *dual biquaternion*;
- For hyperbolic complex numbers with imaginary unit $h^2 = 1, h \neq 0$, the biquaternion is called a *hyperbolic biquaternion*.

A dual biquaternion is sometimes called a Clifford biquaternion, and a hyperbolic biquaternion is sometimes referred to as a Hamilton biquaternion [11]. In english-language sources, a dual biquaternion is more commonly called a *dual quaternion* [10, 20]. In english version of our paper we use the term *dual quaternion* and in russian version we use biquaternion as short name for parabolic biquaternion.

The term motor is a syllabic abbreviation formed from the words **mo**ment and vec*tor*.

4.2. Operations on dual quaternions

4.2.1. Addition

Introduce two dual quaternions P and Q and write them in quaternion and dual form

$$P = p + p^o\varepsilon = P_0 + P_1\mathrm{i} + P_2\mathrm{j} + P_3 k = P_0 + \mathbf{P}, Q = q + q^o\varepsilon = Q_0 + Q_1\mathrm{i} + Q_2\mathrm{j} + Q_3 k = Q_0 + \mathbf{Q}.$$

Addition (and subtraction) is defined by formula:

$$P \pm Q = p \pm q + (p^o \pm q^o)\varepsilon = P_0 \pm Q_0 + (P_1 \pm Q_1)\mathrm{i} + (P_2 \pm Q_2)\mathrm{j} + (P_3 \pm Q_3)\mathrm{k} = P_0 \pm Q_0 + \mathbf{P} \pm \mathbf{Q}.$$

The consequence of this formula is the associativity and commutativity of addition.:

$$P + Q = Q + P \text{ and } P + (Q + R) = (P + Q) + R,$$

where R is some third dual quaternion.

4.2.2. Multiplication by a number

The rule of multiplying a dual quaternion by a real number α is trivial and is determined by the ratio:

$$\alpha Q = \alpha q + \alpha q^o\varepsilon = \alpha Q_0 + \alpha\mathbf{Q}.$$

Multiplication by a dual number $A = \alpha + \alpha^o\varepsilon$ is somewhat more complicated:

$$(\alpha + \alpha^o\varepsilon)(q + q^o\varepsilon) = \alpha q + (\alpha q^o + \alpha^o q)\varepsilon.$$

Similarly, in the dual representation $Q = Q_0 + Q_1\mathrm{i} + Q_2\mathrm{j} + Q_3\mathrm{k}$, $Q_i = q_i + q_i^o\varepsilon$, $i = 0, 1, 2, 3$ yields:

$$AQ_i = (\alpha + \alpha^o\varepsilon)(q_i + q_i^o\varepsilon) = \alpha q_i + (\alpha q_i^o + \alpha^o q_i)\varepsilon.$$

4.2.3. Dual quaternion product

The *dual quaternion product* of P and Q is defined by

$$PQ = (p + p^o\varepsilon)(q + q^o\varepsilon) = pq + (pq^o + p^o q)\varepsilon,$$

where pq, pq^o, and $p^o q$ are quaternion products. In dual representation the formula reproduces the quaternion product, in which all real numbers are replaced by dual components of dual quaternions:

$$PQ = \big(P_0Q_0 - (P_1Q_1 + P_2Q_2 + P_3Q_3)\big) + P_0(Q_1\mathrm{i} + Q_2\mathrm{j} + Q_3\mathrm{k}) + Q_0(P_1\mathrm{i} + P_2\mathrm{j} + P_3\mathrm{k}) + \\ + (P_2Q_3 - P_3Q_2)\mathrm{i} + (P_3Q_1 - P_1Q_3)\mathrm{j} + (P_1Q_2 - P_2Q_1)\mathrm{k}.$$

This expression is conveniently written in the more compact form:

$$PQ = P_0 Q_0 - (\mathbf{P}, \mathbf{Q}) + P_0 \mathbf{Q} + Q_0 \mathbf{P} + \mathbf{P} \times \mathbf{Q},$$

where $(\mathbf{P}, \mathbf{Q}) = P_1 Q_1 + P_2 Q_2 + P_3 Q_3$ is the scalar product, and

$$\mathbf{P} \times \mathbf{Q} = (P_2 Q_3 - P_3 Q_2)\mathrm{i} + (P_3 Q_1 - P_1 Q_3)\mathrm{j} + (P_1 Q_2 - P_2 Q_1)\mathrm{k}$$

is the screw product of pure dual quaternions **P** and **Q**.

For pure dual quaternions:

$$\mathbf{PQ} = -(\mathbf{P}, \mathbf{Q}) + \mathbf{P} \times \mathbf{Q}, \mathbf{QP} = -(\mathbf{P}, \mathbf{Q}) - \mathbf{P} \times \mathbf{Q},$$

$$(\mathbf{P}, \mathbf{Q}) = -\frac{1}{2}\left(\mathbf{PQ} + \mathbf{QP}\right), \mathbf{P} \times \mathbf{Q} = \frac{1}{2}\left(\mathbf{PQ} - \mathbf{QP}\right).$$

4.2.4. Conjugation operations

Since both ordinary (elliptic) and dual complex numbers are present in the definition of a dual quaternion, three conjugation operations are introduced.

- $Q^* = (q + q^o\varepsilon)^* = q^* + q^{o*}\varepsilon$ is quaternion conjugation, which can also be called complex.
- $\overline{Q} = \overline{q + q^o\varepsilon} = q - q^o\varepsilon$ is dual conjugation.
- $Q^\dagger = (\overline{q + q^o\varepsilon})^* = q^* - q^{o*}\varepsilon$ is biquaternion conjugation.

A dual quaternion conjugation is a combination of two other conjugations. The following properties are valid for the introduced conjugation operations:

$$(PQ)^* = Q^* P^*, \overline{QP} = \overline{PQ}, (PQ)^\dagger = Q^\dagger P^\dagger.$$

4.2.5. Scalar product

Scalar product of two arbitrary dual quaternions $P = p + p^o\varepsilon$ and $Q = q + q^o\varepsilon$ is determined by the following formula [20, p. 15]:

$$(P, Q) = \frac{1}{2}(PQ^* + QP^*),$$

where $*$ is quaternion conjugation

$$PQ^* = (p + p^o\varepsilon)(q^* + q^{o*}\varepsilon) = pq^* + [pq^{o*} + p^o q^*]\varepsilon,$$
$$QP^* = (q + q^o\varepsilon)(p^* + p^{o*}\varepsilon) = qp^* + [qp^{o*} + q^o p^*]\varepsilon$$

hence:

$$(P, Q) = \frac{1}{2}(pq^* + qp^*) + \frac{1}{2}(pq^{o*} + q^o p^*)\varepsilon + \frac{1}{2}(p^o q^* + qp^{o*})\varepsilon = (p, q) + [(p, q^o) + (p^o, q)]\varepsilon.$$

where $(p, q) = p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3$, $(p^o, q) = p_0^o q_0 + p_1^o q_1 + p_2^o q_2 + p_3^o q_3$ and $(p, q^o) = p_0 q_0^o + p_1 q_1^o + p_2 q_2^o + p_3 q_3^o$ are quaternion scalar products [20, p. 15], which can be written as $(p, q) = p_0 q_0 + (\mathbf{p}, \mathbf{q})$.

If dual quaternions are written in dual representation as $P = P_0 + \mathbf{P}$ and $Q = Q + \mathbf{Q}$, the formula can be written as follows:

$$PQ^* = (P_0 + \mathbf{P})(Q_0 - \mathbf{Q}) = P_0 Q_0 - P_0 \mathbf{Q} + Q_0 \mathbf{P} + (\mathbf{P}, \mathbf{Q}) - \mathbf{P} \times \mathbf{Q},$$
$$QP^* = (Q_0 + \mathbf{Q})(P_0 - \mathbf{P}) = P_0 Q_0 - Q_0 \mathbf{P} + P_0 \mathbf{Q} + (\mathbf{Q}, \mathbf{P}) - \mathbf{Q} \times \mathbf{Q},$$

$$(P,Q) = \frac{1}{2}(2P_0Q_0 - P_0\mathbf{Q} + Q_0\mathbf{P} - Q_0\mathbf{P} + P_0\mathbf{Q} + 2(\mathbf{P},\mathbf{Q}) - \mathbf{P}\times\mathbf{Q} + \mathbf{P}\times\mathbf{Q}) = P_0Q_0 + (\mathbf{P},\mathbf{Q}),$$

which results in a general formula for the dual quaternion scalar product:

$$(P,Q) = (p,q) + [(p,q^o) + (p^o,q)]\varepsilon = P_0Q_0 + (\mathbf{P},\mathbf{Q}) = \sum_{i=0}^{3} P_iQ_i.$$

Some simple but important consequences can be immediately derived from the definition of the scalar product.

- The scalar product is a dual number.
- The scalar product of dual quaternions is *symmetric* $(P,Q) = (Q,P)$, since the scalar products of quaternions are symmetric and $(p,q) + [(p,q^o) + (p^o,q)]\varepsilon = (q,p) + [(q,p^o) + (q^o,p)]\varepsilon$.
- The scalar product is *bilinear*. So, for dual quaternions P, Q, R and dual numbers α, β the equalities are fulfilled:

  $$(P,\alpha Q + \beta R) = \alpha(P,Q) + \beta(P,R), (\alpha P + \beta Q, R) = \alpha(P,R) + \beta(Q,R).$$

  Indeed:

  $$(P,\alpha Q + \beta R) = \frac{1}{2}\big(P(\alpha Q + \beta R)^* + (\alpha Q + \beta R)P^*\big) = \frac{1}{2}(\alpha PQ^* + \beta PR^* + \alpha QP^* + \beta RP^*) = \\ = \frac{1}{2}\alpha(PQ^* + QP^*) + \frac{1}{2}\beta(PR^* + RP^*) = \alpha(P,Q) + \beta(P,R),$$

  and the second identity follows from symmetry.
- Written in quaternion form, the dual quaternion scalar product reproduces the dual number multiplication rule; written in dual form, it reproduces the quaternion scalar product formula.
- If $P = 0 + \mathbf{P}$ and $Q = 0 + \mathbf{Q}$ are pure dual quaternions, then:

  $$(\mathbf{P},\mathbf{Q}) = (\mathbf{p},\mathbf{q}) + [(\mathbf{p},\mathbf{q}^o) + (\mathbf{p}^o,\mathbf{q})]\varepsilon,$$

  where $(\mathbf{p},\mathbf{q}^o) + (\mathbf{p}^o,\mathbf{q}) = \mathrm{mom}(\mathbf{P},\mathbf{Q})$ is the mutual moment of $\mathbf{P}$ and $\mathbf{Q}$.
- If $P = p + 0\varepsilon$, $Q = q + 0\varepsilon$, then the scalar product of dual quaternions reduces to the scalar product of quaternions $(P,Q) = (p,q)$.
- If $P = p^o\varepsilon$ and $Q = q^o\varepsilon$, despite the fact that both dual quaternions are nonzero, their scalar product turns to zero: $(P,Q) = (0 + \mathbf{0}, 0 + \mathbf{0}) + [(0 + \mathbf{0}, p^o) + (0 + \mathbf{0}, q^o)]\varepsilon = 0$.

For calculations of the scalar product of dual quaternions written in quaternion form, calculations by hand are often easier “on paper” if the terms are slightly rearranged.

$$(P,Q) = (p_0 + \mathbf{p} + p_0^o\varepsilon + \mathbf{p}^o\varepsilon, q_0 + \mathbf{q} + q_0^o\varepsilon + \mathbf{q}^o\varepsilon) = p_0q_0 + (\mathbf{p},\mathbf{q}) + [p_0q_0^o + (\mathbf{p},\mathbf{q}^o) + p_0^oq_0 + (\mathbf{p}^o,\mathbf{q})]\varepsilon,$$

which leads to:

$$(P,Q) = p_0q_0 + [p_0q_0^o + p_0^oq_0]\varepsilon + (\mathbf{p},\mathbf{q}) + [(\mathbf{p},\mathbf{q}^o) + (\mathbf{p}^o,\mathbf{q})]\varepsilon. \tag{6}$$

The concept of *orthogonality* of two dual quaternions can be introduced as follows: two dual quaternions $P = p + \varepsilon p^o$ and $Q = q + \varepsilon q^o$ are called orthogonal if $(p,q) = 0$, which is equivalent to the condition of orthogonality of their main parts.

Consider a special case of basis quaternions, for example, $P = \mathrm{i}$ and $Q = \mathrm{j}$, then, according to formula (6), hence

$$(P,Q) = 0\cdot 0 + [0\cdot 0 + 0\cdot 0]\varepsilon + (\mathrm{i},\mathrm{j}) + [(\mathrm{i},\mathbf{0}) + (\mathbf{0},\mathrm{j})]\varepsilon = (\mathrm{i},\mathrm{j}) = 0.$$

All members of the formula are specifically listed here to demonstrate how the symbols work. The result is the orthogonality of the basis elements i and j.

Now let $P = \mathrm{i}\varepsilon$ and $Q = \mathrm{j}$, then

$$(P,Q) = 0\cdot 0 + [0\cdot 0 + 0\cdot 0]\varepsilon + (\mathbf{0},\mathrm{j}) + [(\mathbf{0},\mathbf{0}) + (\mathrm{i}\varepsilon,\mathrm{j})]\varepsilon = (\mathrm{i},\mathrm{j})\varepsilon^2 = 0.$$

In the same way, it is possible to prove the mutual orthogonality of the basis elements $\langle 1, \mathrm{i}, \mathrm{j}, \mathrm{k}, \varepsilon, \mathrm{i}\varepsilon, \mathrm{j}\varepsilon, \mathrm{k}\varepsilon\rangle$.

4.2.6. Absolute value of a dual quaternion and angle between dual quaternions

The squared absolute value of a dual quaternion is defined by:

$$|Q|^2 = (Q,Q) = QQ^*,$$

hence the following

$$|Q|^2 = QQ^* = (q + q^o\varepsilon)(q^* + q^{o*}\varepsilon) = qq^* + (qq^{o*} + q^o q^*)\varepsilon = |q|^2 + 2(q,q^o)\varepsilon,$$

where (q,q^o) is the quaternion scalar product of q and q^o. It can be seen from the resulting expression that the square of the absolute value $|Q|^2$ is a dual number.

The *absolute value of a dual quaternion* Q is obtained by taking the square root of the dual number $|Q|^2$:

$$|Q| = \sqrt{QQ^*} = \sqrt{|q|^2 + 2(q,q^o)\varepsilon} = |q| + \frac{(q,q^o)}{|q|}\varepsilon = |q|\left(1 + \frac{(q,q^o)}{|q|^2}\varepsilon\right),$$

in accordance with formula (1) for the square root of a dual number.

The following consequences can be proved.

- $|PQ| = |P||Q|$ так как $(PQ,PQ) = (PQ)(PQ)^* = PQQ^*P^* = P|Q|^2P^* = |Q|^2PP^* = |Q|^2|P|^2$ hence, the desired equality is obtained.
- For any three dual quaternions P, Q, R the following equality holds

$$(PQ,PR) = |P|^2(P,R),$$

which follows from

$$(PQ,PR) = \frac{1}{2}\big(PQ(PR)^* + PR(PQ)^*\big) = \frac{1}{2}(PQR^*P^* + PRQ^*P^*) =$$
$$= P\frac{1}{2}(QR^* + RQ^*)P^* = P(Q,R)P^* = |P|^2(Q,R).$$

The scalar product of the main and moment parts (q,q^o) is called the *invariant of the dual quaternion* [16, p. 71], and the real number $k = \frac{(q,q^o)}{|q|^2}$ is called the *parameter of the dual quaternion*. The absolute value can then be written as $|Q| = |q|(1 + k\varepsilon)$.

For a pure dual quaternion (motor) $\mathbf{Q} = \mathbf{q} + \mathbf{q}^o\varepsilon$ the parameter $k = \frac{(\mathbf{q},\mathbf{q}^o)}{\|\mathbf{q}\|^2}$ is determined by the scalar product of the vector $\mathbf{q}$ with its moment $\mathbf{q}^o$. If $(\mathbf{q},\mathbf{q}^o) = 0$, the motor is called a *screw*. More details of this type of dual quaternions will be discussed below.

Note that the expression for the dual quaternion modulus loses its meaning if the absolute value of the main part is equal to zero $|q| = 0$, since in this case the denominator of the fraction turns to zero. The existence of a singularity at a nonzero value of the square of the absolute value distinguishes a dual quaternion from an ordinary quaternion. The square of the module retains its meaning.

Using the scalar product, the cosine of the *dual angle* between two dual quaternions is defined by:

$$\cos \measuredangle(P, Q) = \cos \Theta = \frac{(P, Q)}{|P||Q|}.$$

Here $\cos \Theta$ is a dual number, and, hence, Θ is also a dual number. The geometric meaning of this angle is discussed later, and note that for pure dual quaternions, the formula takes the form:

$$\cos \Theta = \frac{(\mathbf{P}, \mathbf{Q})}{|\mathbf{P}||\mathbf{Q}|} = \frac{(\mathbf{p}, \mathbf{q}) + \operatorname{mom}(\mathbf{P}, \mathbf{Q})}{\mathbf{PQ}}.$$

Unlike QQ^*, the products $Q\overline{Q}$ and $QQ^\dagger$ are not dual numbers. Consider $QQ^\dagger$

$$QQ^\dagger = (q + q^o\varepsilon)(q^* + q^{o*}\varepsilon) = qq^* + (q^o q^* - qq^{o*})\varepsilon = |q|^2 + (q^o q^* - qq^{o*})\varepsilon,$$

$$q^o q^* - qq^{o*} = q_0^o q_0 - q_0^o \mathbf{q} + q_0 \mathbf{q}^o - \mathbf{q}^o \mathbf{q} - q_0 q_0^o + q^o \mathbf{q}^o - \mathbf{q} q_0^o + \mathbf{q}\mathbf{q}^o = 2(q_0 \mathbf{q}^o - q_0^o \mathbf{q}) + 2\mathbf{q} \times \mathbf{q}^o,$$

$$QQ^\dagger = |q|^2 + [2(q_0 \mathbf{q}^o - q_0^o \mathbf{q}) + 2\mathbf{q} \times \mathbf{q}^o]\varepsilon.$$

For $Q\overline{Q}$:

$$Q\overline{Q} = (q + q^o\varepsilon)(q - q^o\varepsilon) = q^2 + (q^o q - qq^o)\varepsilon = q^2 + 2\mathbf{q}^o \times \mathbf{q}\varepsilon = q_0^2 - (\mathbf{q}, \mathbf{q}) + 2q_0\mathbf{q} + 2\mathbf{q}^o \times \mathbf{q}\varepsilon.$$

4.2.7. Screw product

The *screw* product [22, p. 102] is defined only for pure dual quaternions and coincides with the vector product of vectors with dual components. For $\mathbf{P} = \mathbf{p} + \mathbf{p}^o\varepsilon$ and $\mathbf{Q} = \mathbf{q} + \mathbf{q}^o\varepsilon$ are two pure dual quaternions, then

$$\mathbf{P} \times \mathbf{Q} = \mathbf{p} \times \mathbf{q} + (\mathbf{p} \times \mathbf{q}^o + \mathbf{p}^o \times \mathbf{q})\varepsilon.$$

For dual representation, the screw product formula has already been written above, but it can be rewritten in a more compact form using a formal determinant:

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \end{vmatrix} = (P_2 Q_3 - P_3 Q_2)\mathrm{i} + (P_3 Q_1 - P_1 Q_3)\mathrm{j} + (P_1 Q_2 - P_2 Q_1)\mathrm{k}.$$

The operation $\times$ can be extended by setting $\alpha \times \mathbf{P} = \alpha\mathbf{P}$, for $\alpha \in \mathbb{R}$. This extension allows for more compact notation in some formulas.

4.2.8. Unit dual quaternion and trigonometric representation

A *Unit dual quaternion*, is a dual quaternion $U = u + u^o\varepsilon$ satisfying $(U, U) = 1$ or $|U| = 1$, which is equivalent to $|u|^2 + 2(u, u^o)\varepsilon = 1$, so $|u| = 1$ and $(u, u^o) = 0$. Therefore, a unit dual quaternion has zero parameter and zero invariant.

From $(PQ, PR) = |P|^2(P, R)$, multiplication of two dual quaternions by a unit dual quaternion preserves their scalar product: $(UQ, UP) = |U|^2(P, Q)$. Consequently, the dual angle Θ between dual quaternions is preserved.

Consider a unit dual quaternion U in dual representation:

$$U = U_0 + U_1\mathrm{i} + U_2\mathrm{j} + U_3\mathrm{k}, \quad U_i = u_i + u_i^o\varepsilon, \quad i = 0, 1, 2, 3, \quad |U|^2 = U_0^2 + \|\mathbf{U}\|^2 = 1.$$

By virtue of the fundamental trigonometric identity $\sin^2\Theta + \cos^2\Theta = 1$, which is also true for dual numbers, there is always a dual angle Θ such that

$$U_0^2 = \cos^2\Theta, \|\mathbf{U}\|^2 = \sin^2\Theta \Rightarrow U_0^2 + \|\mathbf{U}\|^2 = 1.$$

The screw part $\mathbf{U}$ of the dual quaternion U need not be a unit pure dual quaternion, but it can be normalized:

$$\mathbf{U} = \|\mathbf{U}\|\Big(\frac{U_1}{\|\mathbf{U}\|}\mathrm{i} + \frac{U_2}{\|\mathbf{U}\|}\mathrm{j} + \frac{U_3}{\|\mathbf{U}\|}\mathrm{k}\Big) = \|\mathbf{U}\|\mathbf{E} = \sin\Theta\mathbf{E},$$

где $\mathbf{E}$ is a unit pure dual quaternion (a unit screw).

Thus, any unit dual quaternion $U = U_0 + U_1\mathrm{i} + U_2\mathrm{j} + U_3\mathrm{k}$ can be written in *trigonometric form*:

$$U = \cos\Theta + \sin\Theta\mathbf{E}, \quad \mathbf{E} = E_1\mathrm{i} + E_2\mathrm{j} + E_3\mathrm{k}, \quad \|\mathbf{E}\| = 1,$$

where the dual angle Θ is also called the *dual argument* of the dual quaternion.

An arbitrary dual quaternion P can always be decomposed into the product of a modulus and a unit dual quaternion as follows:

$$P = P_0 + \mathbf{P} = |\mathrm{P}|\Big(\frac{P_0}{|\mathrm{P}|} + \frac{\mathbf{P}}{|\mathrm{P}|}\Big) = |\mathrm{P}|\Big(\frac{P_0}{|\mathrm{P}|} + \frac{\|\mathbf{P}\|}{|\mathrm{P}|}\mathbf{E}\Big) = |\mathrm{P}|(\cos\Theta + \sin\Theta\mathbf{E}),$$

where $\cos^2\Theta = (P_0/|\mathrm{P}|)^2$ and $\sin^2\Theta = (\|\mathbf{P}\|/|\mathrm{P}|)^2$, and the unit dual quaternion $\cos\Theta + \sin\Theta\mathbf{E}$ is sometimes called the *versor* of the dual quaternion P, and the modulus $|\mathrm{P}|$ is called the tensor (this term will not be used).

4.2.9. Dualization Operation

Introduce an algebraic operation closely related to the principle of duality in projective geometry. This connection will become apparent later; here only algebraic properties are considered.

Define the *dualization operation* $+$ acting on the basis elements by

$$1^+ = \varepsilon, \varepsilon^+ = 1,$$
$$\mathrm{i}^+ = \mathrm{i}\varepsilon, (\mathrm{i}\varepsilon)^+ = \mathrm{i},$$
$$\mathrm{j}^+ = \mathrm{j}\varepsilon, (\mathrm{j}\varepsilon)^+ = \mathrm{j},$$
$$\mathrm{k}^+ = \mathrm{k}\varepsilon, (\mathrm{k}\varepsilon)^+ = \mathrm{k}.$$

Additionally, linearity is required: $(P + Q)^+ = P^+ + Q^+$. Then:

$$P^+ = (p + p^o\varepsilon)^+ = p^o + p\varepsilon.$$

The scalar product of a dualized dual quaternion with itself is given by:

$$(P^+, P^+) = |p^o|^2 + 2(p, p^o)\varepsilon.$$

5. Dual quaternion representation of points, lines and planes

We consider the three-dimensional projective space $\mathrm{P}\mathbb{R}^3$ modeled by the four-dimensional Cartesian space $\mathbb{R}^4$ (see article [26]). In this space, each point can be defined by homogeneous coordinates $(x, y, z \mid w)$, also written as $x : y : z : w$. The fourth coordinate w is called the weight coordinate or *weight*. All points are divided into two classes:

proper points are finite points for which $w \neq 0$, corresponding to affine points P with coordinates $(x/w, y/w, z/w)$;

improper points are points at infinity for which $w = 0$, corresponding to free vectors specifying direction $\mathbf{v} = (v_x, v_y, v_z)$.

The German mathematician A. Möbius (Möbius, August Ferdinand, 1790–1868) introduced a mechanical interpretation of homogeneous coordinates by associating the fourth coordinate not with spatial dimension but with the mass of a point, i.e., an additional attribute inherent to each point in space. Such a point is called a *mass point* [27]. Further, for brevity, we will refer to a point with non-unit w coordinate as a *mass point*, and a point with unit w coordinate as an *affine point*.

5.1. Quaternion representation of points

To each mass point P_w with coordinates $(x, y, z \mid w)$ we assign a dual quaternion of the following form:

$$P = p + p^o\varepsilon = w + \mathbf{p}^o\varepsilon, \quad p = w + \mathbf{0}, \quad p^o = \mathbf{p}^o = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}.$$

The scalar part of the quaternion p consists only of the scalar unit, and the vector part p^o is pure and corresponds to the radius vector of the point $\mathbf{p}^o = (x, y, z)^T$. Here we write the radius vector with the symbol o to emphasize the dual quaternion representation. This symbol in this context does not carry any additional geometric meaning.

To each mass point P_w corresponds an affine point with coordinates $P = (x/w, y/w, z/w)$ or in dual quaternion representation:

$$P = 1 + \frac{1}{w}\mathbf{p}^o\varepsilon = 1 + \frac{x}{w}\mathrm{i}\varepsilon + \frac{y}{w}\mathrm{j}\varepsilon + \frac{z}{w}\mathrm{k}\varepsilon.$$

If $w = 1$, then we get $P = 1 + \mathbf{p}^o\varepsilon$.

Free vectors or improper points, which specify direction in space, can also be represented in dual quaternion form as

$$\mathbf{V} = \mathbf{v}\varepsilon = v_x\mathrm{i}\varepsilon + v_y\mathrm{j}\varepsilon + v_z\mathrm{k}\varepsilon.$$

This is a specific pure dual quaternion $\mathbf{V}$ with zero vector part but non-zero moment part.

If given two points P and Q, then the free vector $\mathbf{PQ}$, which specifies the direction from point P to point Q, can be calculated using dual quaternions as follows:

$$\mathbf{PQ} = Q - P = 1 + \mathbf{q}^o\varepsilon - 1 - \mathbf{p}^o\varepsilon = (\mathbf{q}^o - \mathbf{p}^o)\varepsilon.$$

The notations of quaternions well agree with the notations of points and vectors, so from now on "point P" should be interpreted as "dual quaternion P associated with an affine point".

Note that the dual quaternion $O = w + \mathbf{0} + (0 + \mathbf{0})\varepsilon = w$ is associated with the point of the origin $(0, 0, 0 \mid 1)$, so for any point P we can write $P - O = 1 + \mathbf{p}^o\varepsilon - 1 = \mathbf{p}^o\varepsilon$ and geometrically interpret the result as a free vector $\mathbf{p}^o$. This way of denoting free vectors is accepted in the book [20], however, in our opinion this notation is too cumbersome and we will introduce a more compact notation. We will say that to each point $P = 1 + \mathbf{p}^o\varepsilon$ corresponds a pure dual quaternion $\mathbf{P} = \mathbf{p}^o\varepsilon$ which specifies a free vector, which, when plotted from the origin, will set the point P. The last sentence algebraically expressed in the notation $P = O + \mathbf{P} = 1 + \mathbf{P}$.

If given a mass point $P_w = w + \mathbf{p}^o\varepsilon$, then it will also correspond to a free vector $\mathbf{P} = \dfrac{1}{w}\mathbf{p}^o\varepsilon$, which, when added to the dual quaternion $O = 1$, will again give an affine point $P = 1 + \dfrac{1}{w}\mathbf{p}^o\varepsilon$, which can then be converted into a mass point by multiplying by w:

$$wP = P_w = w + \frac{w}{w}\mathbf{p}^o\varepsilon = w + \mathbf{p}^o\varepsilon.$$

Any dual quaternion P_w, representing a mass point, when multiplied by a real number m will represent the same affine point as before multiplication, because

$$mP_w = mw + m\mathbf{p}^o\varepsilon \leftrightarrow P = wm/wm + m\mathbf{p}^o\varepsilon/mw = 1 + \mathbf{p}^o\varepsilon/w.$$

From this it follows that the dual quaternion representation of an affine point is homogeneous, which is consistent with homogeneous coordinates $x : y : z : w$.

5.2. Operations over points

By associating an algebraic object with a point, we will explain the geometric meaning of some dual quaternion operations. In most cases we will give the geometric interpretation to an affine point, even if the calculations were performed over a mass point.

If given an affine point $P = 1 + \mathbf{p}^o\varepsilon$, then the conjugation operations have the following meaning:

- $P^* = 1 - \mathbf{p}^o\varepsilon$ is central symmetry relative to the origin;
- $\bar{P} = P^* = 1 - \mathbf{p}^o\varepsilon$ is also central symmetry;
- $P^\dagger = P$ is identity transformation.

The dual quaternions $P_w = w_1 + \mathbf{p}^o\varepsilon$ and $Q_w = w_2 + \mathbf{p}^o\varepsilon$ commute under dual quaternion multiplication:

$$P_wQ_w = (w_1 + \mathbf{p}^o\varepsilon)(w_2 + \mathbf{p}^o\varepsilon) = w_1w_2 + (w_1\mathbf{q}^o + w_2\mathbf{p}^o)\varepsilon,$$
$$Q_wP_w = (w_2 + \mathbf{p}^o\varepsilon)(w_1 + \mathbf{p}^o\varepsilon) = w_2w_1 + (w_2\mathbf{p}^o + w_1\mathbf{q}^o)\varepsilon.$$

Therefore $P_wQ_w = Q_wP_w = w_1w_2 + (w_2\mathbf{p}^o + w_1\mathbf{q}^o)\varepsilon$ and $PQ = 1 + (\mathbf{p}^o/w_1 + \mathbf{q}^o/w_2)\varepsilon$. The result of multiplication gave an affine point in Cartesian space, the radius vector of which is the sum of the radius vectors of the two multiplied points.

If we perform the addition of dual quaternions $P_w + Q_w = w_1 + w_2 + (\mathbf{p}^o + \mathbf{q}^o)\varepsilon$, then the geometric meaning is only for the sum of affine points $(P + Q)_w = 2 + (\mathbf{p}^o + \mathbf{q}^o)\varepsilon$, therefore $P + Q = 1 + \frac{(\mathbf{p}^o+\mathbf{q}^o)}{2}\varepsilon$ is the midpoint of the segment PQ.

Let's find the scalar product of two mass points P_w and Q_w

$$(P_w, Q_w) = (w_1+\mathbf{0}+(0+\mathbf{p}^o)\varepsilon, w_2+\mathbf{0}+(0+\mathbf{q}^o)\varepsilon) = w_1w_2+[0\cdot 0+0\cdot 0]\varepsilon+(\mathbf{0},\mathbf{0})+[(\mathbf{0},\mathbf{q}^o)+(\mathbf{0},\mathbf{p}^o)]\varepsilon = w_1w_2,$$

From this it follows that the square of the module of a mass point and an affine point:

$$|P_w|^2 = w^2, \quad |P|^2 = 1.$$

Note that the module of a dual quaternion gives the value of the weight coordinate, not the length of the radius vector. However, using the dualization operation, we can obtain the radius vector of a point:

$$|P_w|^2 = |\mathbf{p}^o|^2 + 2(1 + \mathbf{0}, 0 + \mathbf{p}^o)\varepsilon = |\mathbf{p}^o|^2 = \|\mathbf{p}^o\|^2,$$

And also the norm of a free vector $\mathbf{V} = \mathbf{v}\varepsilon$ as $|\mathbf{V}^+| = \|\mathbf{v}\|$.

5.3. Dual quaternion representation of planes

To uniquely define a plane as in three-dimensional Cartesian space and in three-dimensional projective space, it is sufficient to define a linear equation of the following form:

$$n_xx + n_yy + n_zz + d = 0 \Leftrightarrow (\mathbf{n}, \mathbf{p}) + d = 0,$$

Where $\mathbf{p} = (x, y, z)^T$ is the radius vector of an arbitrary point of the plane, $\mathbf{n} = (n_x, n_y, n_z)$ is the direction vector of the normal to the plane, not necessarily unit, and d is a parameter whose geometric meaning is revealed by the relation $d = \delta\|\mathbf{n}\|$, where δ is the directed distance from the plane to the origin.

The unit normal vector will be denoted as $\hat{\mathbf{n}}$. With its help, the equation of the plane can be written in the simplest form: $(\hat{\mathbf{n}}, \mathbf{p}) + \delta = 0$. However, we also note that any vector $\mathbf{n} = k\hat{\mathbf{n}}$ and parameter $d = \delta\|\mathbf{n}\| = k\delta$ will define the same plane as the pair $[\hat{\mathbf{n}} \mid \delta]$.

The plane given by the pair $[\mathbf{n} \mid d]$ has the following dual quaternion representation:

$$\Pi = n + n^o\varepsilon = \mathbf{n} + d\varepsilon, \text{ that is } n = 0 + \mathbf{n}, \quad n^o = d + \mathbf{0},$$

In this case, in accordance with the reasoning of the previous paragraph, the dual quaternion $k\Pi$, where $k \in \mathbb{R}$ corresponds to the same plane as the dual quaternion Π. In other words, the dual quaternion representation is homogeneous and for any dual quaternion we can perform normalization:

$$\widehat{\Pi} = \frac{1}{\|\mathbf{n}\|}\Pi = \frac{\mathbf{n}}{\|\mathbf{n}\|} + \frac{d}{\|\mathbf{n}\|}\varepsilon = \hat{\mathbf{n}} + \delta\varepsilon.$$

Let's calculate the module for the plane dual quaternion Π:

$$|\Pi|^2 = \Pi\Pi^* = (\mathbf{n} + d\varepsilon)(-\mathbf{n} + d\varepsilon) = -\mathbf{n}\mathbf{n} + d\mathbf{n}\varepsilon - d\mathbf{n}\varepsilon = d^2\varepsilon^2 = -(-(\mathbf{n}, \mathbf{n}) + \mathbf{n} \times \mathbf{n}) = \|\mathbf{n}\|^2 \Rightarrow |\Pi| = \|\mathbf{n}\|.$$

The module $|\Pi|$ allows us to obtain the length of the normal vector of the plane, so the normalization process can be written as:

$$\widehat{\Pi} = \frac{\Pi}{|\Pi|} = \hat{\mathbf{n}} + \delta\varepsilon.$$

From this it follows that the dual quaternion Π is a unit dual quaternion.

Note that unlike the module of a point, the module of a plane dual quaternion equals the norm of a vector, not a scalar parameter. As in the case of a point, applying the dualization operation and calculating:

$$|\Pi^+|^2 = |d + \mathbf{n}\varepsilon|^2 = d^2 \Rightarrow |\Pi^+| = d.$$

Now the module of the dual quaternion equals the scalar parameter. This is easily explained by the fact that when applying the dualization operation Π^+ we get the dual quaternion $d + \mathbf{n}\varepsilon$, which exactly corresponds to the dual quaternion representation of a point with a weight coordinate d and radius vector $\mathbf{n}$. This is an algebraic expression of the fundamental principle of duality of projective geometry.

Let's prove a simple but important statement. The point $P = 1 + \mathbf{p}^o\varepsilon$ belongs to the plane $\Pi = \mathbf{n} + d\varepsilon$ if and only if the equality:

$$(P, \Pi) = 0. \tag{7}$$

To prove this, we use the formula (6) for calculating the scalar product:

$$(P, \Pi) = 1 \cdot 0 + [1 \cdot d + 0 \cdot 0]\varepsilon + (\mathbf{0}, \mathbf{n}) + [(\mathbf{0}, \mathbf{0}) + (\mathbf{p}^o, \mathbf{n})]\varepsilon = (d + (\mathbf{p}^o, \mathbf{n}))\varepsilon,$$

From this it follows that the condition $(P, \Pi) = 0$ is equivalent to the condition $d + (\mathbf{p}^o, \mathbf{n}) = 0$, i.e., the point P satisfies the equation of the plane and therefore belongs to this plane.

Let's also prove that if a point $P = 1 + \mathbf{p}^o\varepsilon$ and a vector $\mathbf{n}$ are given, then the plane with the normal vector $\mathbf{n}$, passing through the point P, is given by the following equality:

$$\Pi = \mathbf{n} - (P, \mathbf{n}).$$

The scalar product of two dual quaternions is understood as follows:

$$(P, \mathbf{n}) = \big(1 + \mathbf{0} + (0 + \mathbf{p}^o)\varepsilon, 0 + \mathbf{n} + (0 + \mathbf{0})\varepsilon\big) = 1 \cdot 0 + \big[(\mathbf{p}^o, \mathbf{n}) + (1 + \mathbf{0}, 0 + \mathbf{0})\big]\varepsilon = (\mathbf{p}^o, \mathbf{n})\varepsilon.$$

Let's also calculate the scalar product $(P, (P, \mathbf{n}))$:

$$\big(P, (P, \mathbf{n})\big) = \big(P, (\mathbf{p}^o, \mathbf{n})\varepsilon\big) = (\mathbf{p}^o, \mathbf{n})(P, \varepsilon) = (\mathbf{p}^o, \mathbf{n})\big((1 + \mathbf{0}, 0 + \mathbf{0}) + [(\mathbf{p}^o, \mathbf{0}) + (1 + \mathbf{0}, 1 + \mathbf{0})]\big)\varepsilon = (\mathbf{p}^o, \mathbf{n})\varepsilon.$$

Now we subtract $(P, \mathbf{n}) - \big(P, (P, \mathbf{n})\big) = \big(P, \mathbf{n} - (P, \mathbf{n})\big) = (\mathbf{p}^o, \mathbf{n})\varepsilon - (\mathbf{p}^o, \mathbf{n})\varepsilon = 0$ from where, according to condition (7), it follows that the expression $\mathbf{n} - (P, \mathbf{n})$ must be a dual quaternion representing a plane, because the point P belongs to the plane by the condition of the problem.

The condition (7) also allows us to prove that if two points P and Q belong to a plane, then the free vector $\mathbf{PQ}$ also belongs to it. To prove this, we need to calculate

$$(\mathbf{PQ}, \Pi) = (Q - P, \Pi) = (Q, \Pi) - (P, \Pi) = 0.$$

We also show that the projection of the origin O onto the plane Π is given by the point $O' = 1 + \delta\hat{\mathbf{n}}\varepsilon$. This is obvious from the above reasoning, but we will prove it using condition (7). We write:

$$(O', \Pi) = \big(1 + \mathbf{0} + (0 + \delta\hat{\mathbf{n}})\varepsilon, 0 + \mathbf{0} + (d + \mathbf{0})\varepsilon\big) =$$
$$(1 + \mathbf{0}, 0 + \mathbf{n}) + \big[(1 + \mathbf{0}, d + \mathbf{0}) + (0 + \delta\hat{\mathbf{n}}, 0 + \mathbf{n})\big]\varepsilon = 0 + [d + \delta(\mathbf{n}, \hat{\mathbf{n}})]\varepsilon.$$

Now the condition $(O', \Pi) = 0$ is equivalent to $d + \delta(\mathbf{n}, \hat{\mathbf{n}}) = 0$ or $d = \delta\|\mathbf{n}\|(\hat{\mathbf{n}}, \hat{\mathbf{n}}) = \delta\|\mathbf{n}\|$, but this is the geometric meaning of the parameter d, so the point O' indeed belongs to the plane.

Let's prove another formula. There are 8 possible mutual arrangements of three planes:

$$\Pi_i = \mathbf{n}_i - (\mathbf{p}_i^o, \mathbf{n})\varepsilon, \qquad i = 1, 2, 3.$$

One of these arrangements is the intersection of the planes in a single point. The dual quaternion corresponding to this point can be calculated by the formula [20, p. 23, 28, p. 56]:

$$P = 1 + \frac{(\mathbf{n}_1, P_1)\mathbf{n}_2 \times \mathbf{n}_3 + (\mathbf{n}_2, P_2)\mathbf{n}_3 \times \mathbf{n}_1 + (\mathbf{n}_3, P_3)\mathbf{n}_1 \times \mathbf{n}_2}{(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)}.$$

To prove this, we need to use the same condition (7) of belonging of a point to a plane by sequentially multiplying P by Π_1, Π_2, Π_3 and verifying the condition.

A line, or more precisely an axis (a line with a specified direction on it) is represented by a pure dual quaternion of the following form:

The coordinates of the Plücker line, which are the six components $\{\mathbf{v} \mid \mathbf{m}\} = \{v_x, v_y, v_z \mid m_x, m_y, m_z\}$, were introduced by the German physicist and mathematician Julius Plücker in a way that is completely unrelated to dual quaternions. The condition $(\mathbf{v}, \mathbf{m}) = 0$ was also obtained by Plücker and is known as the *Plücker condition*.

A straight line, or rather an axis (a straight line with the direction indicated on it) is given by a pure dual quaternion of the following form:

$$\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon,$$

with the mandatory condition $(\mathbf{v}, \mathbf{m}) = 0$. The vector $\mathbf{v} = v_x\mathrm{i} + v_y\mathrm{j} + v_z\mathrm{k}$ is the guiding vector of the straight line, the vector $\mathbf{m} = m_x\mathrm{i} + m_y\mathrm{j} + m_z\mathrm{k}$ is called the *moment* of the straight line.

Using the dualization operation allows us to rewrite many formulas in dual quaternion notation. For example, let's say we have a dual quaternion point $P = 1 + \mathbf{p}\varepsilon = 1 + \mathbf{P}$ and a pure dual quaternion representing a free vector $\mathbf{V} = \mathbf{v}\varepsilon$. We define the moment of the line passing through the point P in the direction $\mathbf{V}$ by the following formula:

This formula completely coincides with the vector definition of the moment. This notation allows us to define the dual quaternion form of a line as the sum of

$$\mathbf{M} = \mathbf{P} \times \mathbf{V}^{+} = (0 + \mathbf{0}) \times (0 + \mathbf{v}) + [(0 + \mathbf{0}) \times (0 + \mathbf{0}) + (0 + \mathbf{p}) \times (0 + \mathbf{v})]\varepsilon = \mathbf{p} \times \mathbf{v}\varepsilon = \mathbf{m}\varepsilon.$$

The last form of writing will be valid if we additionally put $\alpha \times \mathbf{V}^{+} = \alpha\mathbf{V}^{+}$, where $\alpha \in \mathbb{R}$

$$\mathbf{L} = \mathbf{V}^{+} + \mathbf{M} = \mathbf{V}^{+} + \mathbf{P} \times \mathbf{V}^{+} = (1 + \mathbf{P}) \times \mathbf{V}^{+} = P \times \mathbf{V}.$$

The points of a straight line can be calculated using a parametric representation of a straight line in dual quaternion form:

$$P(t) = P_0 + \mathbf{v}t\varepsilon, \quad P_0 = 1 + \frac{\mathbf{v} \times \mathbf{m}}{\|\mathbf{v}\|^2}\varepsilon.$$

The nearest point to the origin on the line is calculated by the formula:

$$\mathbf{p}_0 = \frac{\mathbf{v} \times \mathbf{m}}{\|\mathbf{v}\|^2}.$$

The dual object to a line is another line: $\mathbf{L}^{+} = (\mathbf{v} + \mathbf{m}\varepsilon)^{+} = \mathbf{m} + \mathbf{v}\varepsilon = \mathbf{M}^{+} + \mathbf{V}$, which is consistent with the principle of duality of projective geometry.

Since a line is represented by a pure dual quaternion, we will say not about the module, but about the norm of a pure dual quaternion. We calculate the norm of the line $\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon$ having in mind the Plücker condition $(\mathbf{v}, \mathbf{m}) = 0$.

$$\mathbf{L}\mathbf{L}^{*} = -(\mathbf{v}+\mathbf{m}\varepsilon)(\mathbf{v}+\mathbf{m}\varepsilon) = -(\mathbf{v}\mathbf{v}+\mathbf{v}\mathbf{m}\varepsilon+\mathbf{m}\mathbf{v}\varepsilon) = -\Big(-(\mathbf{v},\mathbf{v})+\mathbf{v}\times\mathbf{v}-\big((\mathbf{v},\mathbf{m})+\mathbf{v}\times\mathbf{m}-(\mathbf{m},\mathbf{v})+\mathbf{m}\times\mathbf{v}\big)\varepsilon\Big) = \\ = -(-\|\mathbf{v}\|^2 - 2(\mathbf{v},\mathbf{m})\varepsilon) = \|\mathbf{v}\|^2 \quad \Rightarrow \quad \mathbf{L}\mathbf{L}^{*} = \|\mathbf{L}\|^2 = \|\mathbf{v}\|^2.$$

The normalization process for a line reduces to dividing the vectors **v** and **m** by the norm $\|\mathbf{v}\|$.

5.4. Calculation of distances

For dual quaternions representing points $P_w = w + \mathbf{p}\varepsilon$, lines $\mathbf{L} = \mathbf{v} + \varepsilon\mathbf{m}$ and planes $\Pi = \mathbf{n} + d\varepsilon$ we have found formulas for their scalar products:

$$(P_w, P_w) = w^2, \quad (L, L) = \|\mathbf{v}\|^2, \quad (\Pi, \Pi) = \|\mathbf{n}\|^2.$$

Now let's find the scalar products of the dual objects

$$(P_w^{+}, P_w^{+}) = \|\mathbf{p}\|^2, \quad (L^{+}, L^{+}) = \|\mathbf{m}\|^2, \quad (\Pi^{+}, \Pi^{+}) = d^2.$$

It turns out that the formulas for calculating the distance δ from the origin O to the point P_w, the line **L** and the plane Π will have the same form for each of these objects:

$$\delta(O, P_w) = \frac{|P_w^{+}|}{|P_w|}, \quad \delta(O, \mathbf{L}) = \frac{\|\mathbf{L}^{+}\|}{\|\mathbf{L}\|}, \quad \delta(O, \Pi) = \frac{|\Pi^{+}|}{|\Pi|}.$$

6. Results

The main results of the work are summarized in several tables.

- In the table 3, the notation of points, vectors, lines, and planes from different geometries is compared: Euclidean, projective, and dual quaternion representations of projective geometry.
- Table 4 is the main result of our work. The table from the book [29] was taken as a basis (a shorter version of it is also in the book [30]), which has been completely rewritten in the dual quaternion formalism.
- The table 5 contains formulas for normalization of dual quaternions representing points, vectors, lines and planes.
- The table 6 contains a summary of terms that actually refer to the same geometric entity.

As far as the authors can tell, most of the formulas from the table 4 are original results and are not found in publications, although their vector and geometric algebra variants are known.

6.1. Tables

Table 3

Comparison of algebraic representations of points, lines and planes

Geometric object	**Biquaternion representation**	**Homogeneous coordinates**	**Three-dimensional Cartesian space**
Affine point	$P = 1 + \mathbf{p}\varepsilon$,		
$\mathbf{p} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$	$\vec{\mathbf{p}} = (\mathbf{p} \mid 1) = (x, y, z \mid 1)$	$\mathbf{p} = (x, y, z)^T$	
Point mass	$P = w + \mathbf{p}\varepsilon$	$\vec{\mathbf{p}} = (\mathbf{p} \mid w) = (x, y, z \mid w)$	$\mathbf{p} = (x/w, y/w, z/w)$
Vector	$\mathbf{V} = \mathbf{v}\varepsilon$,		
$\mathbf{v} = v_x\mathrm{i} + v_y\mathrm{j} + v_z\mathrm{k}$	$\vec{\mathbf{v}} = (\mathbf{v} \mid 0) = (v_x, v_y, v_z \mid 0)$	$\mathbf{v} = (v_x, v_y, v_z)^T$	
Line	$\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon$	$\vec{\mathbf{L}} = \{\mathbf{v} \mid \mathbf{m}\}$ $\vec{\mathbf{p}} = (\mathbf{v} \times \mathbf{m} \mid \Vert\mathbf{v}\Vert^2)$	$\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{v}t$
Plane	$\Pi = \mathbf{n} + d\varepsilon$, $\mathbf{n} = n_x\mathrm{i} + n_y\mathrm{j} + n_z\mathrm{k}$	$\vec{\ } = [\mathbf{n} \mid d]$	$ax + by + cz + d = 0$

Table 4

Summary of biquaternion formulas for points, lines, and planes

	Formula	**Description**
A	$(Q_w - P_w)^+ + \mathbf{P} \times \mathbf{Q}^+$	Line through two point masses P_w and Q_w
B	$(Q - P)^+ + \mathbf{P} \times \mathbf{Q}^+$	Line through two affine points P and Q
C	$\mathbf{V}^+ + \mathbf{P} \times \mathbf{V}^+$	Line defined by free vector $\mathbf{V}$ and affine point P
D	$(P - O)^+ + \mathbf{0}\varepsilon$	Line through the origin $O = 1$ and point P

Continued on next page

Table 4

Summary of biquaternion formulas for points, lines, and planes (Continued)

	Formula	Description
E	$(\mathbf{V}^+ \times \mathbf{P} - w\mathbf{M})^+ - (\mathbf{L}^+, P_w)$	Plane containing line **L** and point mass P_w
F	$(\mathbf{V}^+ \times \mathbf{P})^+ - (\mathbf{L}^+, P)$	Plane containing line **L** and affine point P
G	$(\mathbf{V}^+ \times \mathbf{U})^+ - (\mathbf{L}^+, \mathbf{U})$	Plane containing line **L** and free vector **U**
H	M^+	Plane containing a line and the origin
I	$\Pi_1 \times \Pi_2$	Line of intersection of two planes
J	$\mathbf{L}^+ \times \Pi^+ - d\mathbf{M}^+ - (\Pi^+, \mathbf{L})$	Point mass of the intersection of line and plane
K	$P \times \Pi - (P, \Pi^+)^+$	Line through the point mass P_w perpendicular to line L
L	$(\mathbf{V} \times \Pi)^+ - (\Pi^+, \mathbf{L}^+)$	Plane containing line L, and perpendicular to plane Π
M	$(\mathbf{V} \times \mathbf{P})^+ - (\mathbf{L}, P)$	Plane containing point mass P_w, and perpendicular to line L
N	$(\mathbf{L}, \mathbf{L}) + (\mathbf{L} \times \mathbf{L}^+)^+$	Point mass on line L, the closest to the origin
O	$(\Pi, \Pi^+) - \Pi \times \Pi^+$	Point mass on plane Π, the closest to the origin
P	$(\mathbf{L}^+, \mathbf{L}^+)^+ + \mathbf{L}^+ \times \mathbf{L}$	The farthest plane from the origin containing line **L**
Q	$(P, P^+) - P \times P^+$	The farthest plane from the origin containing point P
R	$\dfrac{\lvert P_w\rvert Q_w - \lvert Q_w\rvert P_w}{(P_w, Q_w)}$	Distance between point masses
S	$\dfrac{\lvert(\mathbf{L}_1, \mathbf{L}_2)^+\rvert}{\lVert\mathbf{L}_1 \times \mathbf{L}_2\rVert}$	Distance between lines
T	$\dfrac{\lVert(P_w \times \mathbf{L})^+\rVert}{\lvert P_w\rvert \lVert\mathbf{L}\rVert}$	Distance from line to point mass P_w
U	$\dfrac{\lVert\mathbf{L}^+\rVert}{\lVert\mathbf{L}\rVert}$	Distance from line to the origin
V	$\dfrac{\lvert(\Pi, P_w)^+\rvert}{\lvert P_w\rvert \lvert\Pi\rvert}$	Distance from plane to point mass P_w
W	$\dfrac{\lvert\Pi^+\rvert}{\lvert\Pi\rvert}$	Distance from plane to the origin
X	$\dfrac{\lvert P_w^+\rvert}{\lvert P_w\rvert}$	Distance from point mass to the origin

Remark: Point mass $P_w = w + \mathbf{p}\varepsilon$, affine point $P = 1 + \mathbf{p}\varepsilon$, free vector $\mathbf{V} = \mathbf{v}\varepsilon$, line $\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon = \mathbf{V}^+ + \mathbf{M}$, plane $\Pi = \mathbf{n} + d\varepsilon$.

Table 5
Normalization of points, lines, and planes

	General form	Normalized form
Point mass	$P_w = w + \mathbf{p}\varepsilon$	$\hat{P} = \dfrac{P_w}{\lvert P_w \rvert} = 1 + \dfrac{1}{w}\mathbf{p}\varepsilon$
Line	$\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon$	$\hat{\mathbf{L}} = \dfrac{\mathbf{L}}{\lVert \mathbf{L} \rVert} = \dfrac{\mathbf{v}}{\lVert \mathbf{v} \rVert} + \dfrac{\mathbf{m}}{\lVert \mathbf{v} \rVert}\varepsilon$
Plane	$\Pi = \mathbf{n} + d\varepsilon$	$\hat{\Pi} = \dfrac{\Pi}{\lvert \Pi \rvert} = \dfrac{\mathbf{n}}{\lVert \mathbf{n} \rVert} + \dfrac{d}{\lVert \mathbf{n} \rVert}\varepsilon$

Table 6
Correlation of terms of different geometries of space

Affine point	Vector	Point mass	Pure Biquaternion
- finite point - proper point - position vector - bound vector - vector point - quaternion with $q_0 = 1$	- point at infinity - improper point - free vector - direction vector - pure quaternion	- finite point with $w \neq 1$ - quaternion with $q_0 \neq 1$	- screw with zero parameter - dual vector - dyad - null system

7. Conclusion

We have consistently outlined the basics of the algebra of dual numbers, quaternions, and dual quaternions. We have considered the dual quaternion representation of points, lines and planes. The issues of using dual quaternions to describe helical motion were beyond the scope of consideration. We plan to consider these issues in the future. The topic of using dual quaternions to describe helical motion is voluminous, since almost all publications on the topic of dual quaternions are devoted to it. This is because this area is the main, if not the only, application area for dual quaternions.

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Бикватернионное представление точек, прямых и плоскостей

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Аннотация. *Предпосылки* Основная масса работ по бикватернионам, посвящена их применению для описания винтового движения. Представлению с их помощью точек, прямых и плоскостей (примитивов) уделяется мало внимания. *Цель* Необходимо последовательно изложить бикватернионную теорию представления примитивов и доработать математический формализм. *Методы* Используется алгебра дуальных чисел, кватернионов и бикватернионов, а также элементы теории винтов и скользящих векторов. *Результаты* Получены и систематизированы формулы которые использую исключительно бикватернионные операции и обозначения для решения стандартных задач трёхмерной геометрии. *Выводы* Бикватернионы могут служить полноценным формализмом алгебраического представления трёхмерного проективного пространства.

Ключевые слова: дуальные числа, кватернионы, дуальные кватернионы, проективное пространство