HYDRODYNAMIC INSTABILITY OF SPATIALLY PERIODIC FLOWS OF HOMOGENEOUS AND STRATIFIED FLUID WITH REGARD FOR FRICTION. FORMATION OF STEADY-STATE VORTEX DISTURBANCES

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Abstract. The stability of spatially periodic flows of homogeneous and stratified fluid is investigated with regard for bottom friction. The Galerkin method with three basis Fourier harmonics is used to solve the stability problem. A system of ordinary differential equations for the amplitudes of the Fourier harmonics is formulated. A solution to the linearized version of the system is obtained and an expression for the increment of disturbance growth is found. It is established that at the nonlinear stage of development the exponential growth of linear disturbances is replaced by the regime of establishing steady-state periodic disturbances in form of closed cells. These disturbances reduce the averaged horizontal velocity of the flow. Analytical expressions for the spatial period and amplitude of steady-state disturbances are obtained.

Keywords: *hydrodynamic instability, bottom friction, growth rate, vortex cells*

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INTRODUCTION

Extensive literature is devoted to research of hydrodynamic instability of geophysical currents (a large list of publications in [1-3]). The development of instability is associated with the formation of atmospheric cyclones, synoptic vortices in the ocean, ordered vortex tracks, etc.

In this work, the stability of spatially periodic flows of homogeneous and stratified fluid taking into account bottom friction is investigated. The Galerkin method with three basis Fourier-harmonics is used for the solution of the stability problem. A system of ordinary differential equations for the amplitudes of the Fourier-harmonics is formulated. On the basis of the numerical solution of this system, it was shown in previous works that in the frictionless model the development of instability of flows leads to the establishment of a regime of unsteady oscillations or shafts. In this work it is shown that the situation changes fundamentally when friction is taken into account. Exponential growth of linear perturbations at the nonlinear stage of development is replaced by the regime of establishment of stationary periodic perturbations in space. These perturbations decrease the averaged horizontal flow velocity. Analytical expressions for the spatial period and amplitude of perturbations are obtained.

1. PERIODIC FLOW IN A MODEL OF A MONOTONOUS FLUID

We consider two-dimensional motions of a homogeneous incompressible viscous fluid with characteristic horizontal scale D and velocity U_0 in the presence of an external force f . The dimensionless variables of the motion are described by the equation

$$(\Delta\psi)_t + \psi_x(\Delta\psi)_y - \psi_y(\Delta\psi)_x = R^{-1}\left(\Delta^2\psi + f\right). \qquad (1.1)$$

Here ψ is the current function, ,$u = -\psi_y v = \psi_x$ are velocity components, $R = U_0 D/\nu$ is the Reynolds number, ν is the kinematic toughness coefficient, Δ – is the Laplace operator. As scales of coordinates, time and current function are taken respectively D, D/U_0, $U_0 D$, amplitude of external force $f_0 = \nu U_0 / D^2$.

The system of equations (1.1) has an exact solution

$$\psi = \Psi(y) = \sin y, u = U(y) = -\cos y, \qquad (1.2)$$

describing a stationary spatially periodic flow excited by an external periodic force $f = -\sin y$. This flow is called "Kolmogorov flow". The flow (1.2) is an exact solution of (1.1) and in the absence of toughness $R^{-1} = 0$ (non-viscous Kolmogorov flow).

Currents (1.2) are reproduced quite well by the magnetohydrodynamic method in laboratory experiments with a conducting fluid [4-7]. Analogs of such currents can also be created in natural conditions, for example, under the action of periodic distribution of wind tangential voltage in the ocean or external heat inflows in the atmosphere.

The problem on the stability of the flow (1.2) in a viscous fluid was first posed by A. N. Kolmogorov in 1960. A year later, a linear version of the problem obtained a solution in the famous paper [7]. For research of linear stability in this work, the apparatus of the chain fraction theory was used to determine the minimum critical value of the Reynolds number and the wavelength of the most dangerous perturbation. The high degree of symmetry and the possibility of laboratory modeling of the Kolmogorov flow (1.2) stimulated numerous theoretical researches of its stability in the presence of a series of complicating factors (friction, beta effect, stratification) [8-15]. A long-wave approach to the research of nonlinear stability of the Kolmogorov flow using integral solvability conditions for periodic perturbations was proposed in [16, 17]. Using this approach, a weakly nonlinear stability theory was developed that is valid for small deviations of the Reynolds number R from the critical value. A review of publications devoted to research on the stability of the Kolmogorov flow is presented in [4] and article [22].

The linear dynamics of perturbations at large values of the Reynolds number was studied in the mentioned article [7] and in [15]. The nonlinear dynamics of perturbations of Kolmogorov flow in the absence of friction ($R = \infty$) was studied in recent papers [15, 17]. In this work, we consider an approach to describe the nonlinear dynamics of perturbations at finite values of the Reynolds number. This approach uses the Galerkin method with three basis trigonometric functions and time-dependent perturbation amplitudes. It is shown that the time behavior of the perturbation amplitudes is described by a system of three nonlinear differential equations. It is found that in the model with friction the exponential growth of linear perturbations of the Kolmogorov flow is replaced by the regime of establishment of stationary periodic perturbations in the form of closed cells. These perturbations lead to meandering of the flow and reduce its mean velocity. Analytical expressions for the spatial period and amplitude of perturbations are obtained.

2. GALERKIN METHOD AND LINEAR STABILITY THEORY

Representing in (1.1) $\psi = \Psi(y) + \psi'$, to describe the dynamics of small perturbations of ψ' stationary non-viscous flow (1.2) we obtain the equation

$$(\Delta\psi')_t - (\Delta\psi' + \psi')_x \cos y = R^{-1}\Delta^2\psi'. \qquad (2.1)$$

In [7] exponentially increasing with time and periodic in coordinate y solutions of equation (2.1), represented by a series of trigonometric functions $\cos(ny), \sin(ny)$ were found. Finding the increment of acceleration was reduced to the analysis of a rather cumbersome algebraic equation containing an infinite chain fraction.

A simplified approach to constructing solutions to equation (2.1) is based on the Galerkin method with three basis functions $f_1(y) = 1$, $f_2(y) = \cos y$, $f_3(y) = \sin y$ (the first three terms of trigonometric series). We denote by angle brackets $\langle \phi \rangle = (2\pi)^{-1} \int_0^{2\pi} \phi \, dy$ the operation of averaging over the period 2π of the background flow and write equation (2.1) in symbolic form $L(\psi') = 0$. In accordance with the Galerkin method, we find an approximate solution of equation (2.1) in the form (we omit the dash at perturbations)

$$\psi = A(x,t)\cos y + B(x,t)\sin y + C(x,t). \qquad (2.2)$$

We find the expansion coefficients from the orthogonality conditions , $\langle f_i(y) L(\psi) \rangle = 0 i = 1,2,3$. Thus, we obtain a system of equations for the expansion coefficients

$$\left(A_{xx} - A\right)_t - \left(C_{xx} + C\right)_x = R^{-1}\left(A_{xxxx} - 2A_{xx} + A\right), \qquad C_t - (1/2) A_x = R^{-1} C_{xx} . \quad (2.3)$$

These equations are joined by an isolated equation

$$\left(B_{xx} - B\right)_t = R^{-1}\left(B_{xxxx} - 2B_{xx} + B\right), \qquad (2.4)$$

having only damped solutions over time, in particular $B = 0$.

Finding solutions to equations (2.3) of the form $A = a_0 e^{\lambda t} \cos(kx)$, $C = c_0 e^{\lambda t} \sin(kx)$ leads to the equation for the square for the incremental rise λ

$$\left(k^2 + 1\right)\lambda^2 + R^{-1}\left(k^2 + 1\right)\left(2k^2 + 1\right)\lambda + R^{-2}k^2\left(k^2 + 1\right)^2 + (1/2)k^2\left(k^2 - 1\right) = 0. \qquad (2.5)$$

Equating the free term of equation (2.5) to zero, we obtain the equation of the neutral stability curve on the parameter plane (k, R) . This curve separates the growing solutions $(\lambda > 0)$ from the damped solutions $(\lambda < 0)$ and is described by the equation

$$R = R_{cr}(k) = \sqrt{2}\frac{k^2 + 1}{\sqrt{1 - k^2}} . \qquad (2.6)$$

Formula (2.6) gives an excellent approximation of the neutral curve equation obtained by the chain fraction theory method in [7]. According to this formula, instability exists if $R > \sqrt{2}$, and the interval of wave numbers of unstable modes lies inside the interval $0 < k < 1$.

Note that according to (2.5) , in the absence of friction ($R^{-1} = 0$) the square of the accretion increment $\lambda^2 = (1/2)k^2\left(1 - k^2\right)/\left(1 + k^2\right)$. According to this expression, the long-wave perturbations with $0 < k^2 < 1$ are exponentially increasing. The most dangerous perturbation with the maximum increment corresponds to the wave number $k = k_m = \sqrt{\sqrt{2} - 1} \approx 0{,}64$. The dynamics of linear and nonlinear perturbations in the absence of friction has been studied in detail in our recent works [18, 20].

3. NONLINEAR DYNAMICS OF PERTURBATIONS OF PERIODIC FLOW OF HOMOGENEOUS FLUID

The results of the linear analysis suggest that the Galerkin method will give a good approximation of solutions in the nonlinear case as well. For the flow (1.2), the nonlinear dynamics of perturbations is described by Eq.

$$(\Delta\psi')_t - \left(\Delta\psi' + \psi'\right)_x \cos y + \psi'_x(\Delta\psi')_y - \psi'_y(\Delta\psi')_x = R^{-1}\Delta^2\psi' \qquad (3.1)$$

We will look for an approximate solution of (3.1) in the form (2.2). Substituting (2.2) into (3.1) and using the orthogonality conditions to the system of functions $f_i(y)$, we obtain the following nonlinear equation system of partial differential equations for determining the expansion coefficients

$$\left(A - A_{xx}\right)_t + \left(C_{xx} + C\right)_x + \left(B - B_{xx}\right)C_x + BC_{xxx} = -R^{-1}\left(A_{xxxx} - 2A_{xx} + A\right),$$
$$\left(B - B_{xx}\right)_t - \left(A - A_{xx}\right)C_x - AC_{xxx} = -R^{-1}\left(B_{xxxx} - 2B_{xx} + B\right),$$
$$C_t - (1/2)A_x - (1/2)\left(BA_x - AB_x\right) = R^{-1}C_{xx}. \qquad (3.2)$$

Here the first two equations (3.2) are multiplied by -1 for simplicity. The linear version of the system (3.2) reduces to (2.3), (2.4). Note that for the variable C the Galerkin method gives the equation $C_{xxt} - (1/2)\left((B+1)A_x - A(B+1)_x\right)_{xx} = R^{-1}C_{xxxx}$. The third equation (3.2) is obtained from this equation by lowering the order.

Assuming $B = B_1 - 1$ and considering that $B_1 = B_1(t)$ depends only on time, let us write the system (3.2) in the form of

$$\left(A - A_{xx}\right)_t + B_1\left(C_{xx} + C\right)_x = -R^{-1}\left(A_{xxxx} - 2A_{xx} + A\right),$$
$$(B_1)_t - \left(A - A_{xx}\right)C_x - AC_{xxx} = -R^{-1}\left(B_1 - 1\right),$$
$$C_t - (1/2)B_1(t)A_x = R^{-1}C_{xx} \qquad (3.3)$$

Now we assume

$$A = a(t)\cos(kx),\ C = c(t)\sin(kx),\ B_1 = b(t)$$

Substituting these expressions into (3.3) leads to a system of nonlinear ordinary differential equations with respect to amplitudes $a = a(t), c = c(t), b = b(t)$

$$a_t(t)\left(1 + k^2\right) + bc(t)k\left(1 - k^2\right) = -R^{-1}\left(k^2 + 1\right)^2 a,$$
$$b_t - (1/2)kac = -R^{-1}(b - 1),$$
$$c_t + (1/2)kab(t) = -R^{-1}k^2 c\,. \qquad (3.4)$$

Here the lower letter indexes denote partial derivatives in time. The mean equation (3.4) follows from the equation $(b_1)_t - \left(1 + k^2\right) kac\cos^2 x + ack^3 \cos^2 x = 0$, after applying the formulae of degree reduction. The rest of the equations are exact.

The case with absence of friction was considered in our previous works [18, 20]. In this case the system (3.4) is reduced to a nonlinear system

$$a_t(t)(1 + k^2) + bc(t)k(1 - k^2) = 0, \quad b_t - (1/2)kac = 0, \quad c_t + (1/2)kab(t) = 0 \; .$$

The laws of conservation follow from this system

$$\frac{d}{dt}\left(b^2 + c^2\right) = 0 \frac{\mathrm{d}}{\mathrm{d}t}\left(g(k)a^2 + b^2 - c^2\right) = 0 \, ,$$

where $g(k) = \left(1 + k^2\right)/\left(1 - k^2\right)$. Using these laws allows us to represent the solution of the system in terms of elliptic functions. This solution describes nonlinear oscillations or vascillations. An example of the numerical solution of the system (3.4) in the case of $R^{-1} = 0$ for the value k = 0. 5 is presented in Fig.1. As calculations show, at values of R close to zero, damped oscillations with time take place. A completely different behavior is observed in the model with friction at small but finite values of Reynolds number R. In this case, instead of oscillations, the regime of establishment of stationary spatially periodic flows is realized. Let us dwell on this regime in more detail.

It is easy to see that the stationary version of the system (3.4) in the model with friction

$$bck\left(1 - k^2\right) + R^{-1}\left(k^2 + 1\right)^2 a = 0,$$
$$-(1/2)kac + R^{-1}(b - 1) = 0,$$
$$(1/2)kab + R^{-1}k^2 c = 0 \, , \qquad (3.5)$$

has an exact stationary solution. Indeed, from the last equation (3.5) follows

$$c = -\frac{(1/2)kab}{R^{-1}k^2} \; . \qquad (3.6)$$

Substituting this expression into the first equation (3.5) after reduction by a and simple transformations, we obtain $b^2 = 2\frac{R^{-2}\left(k^2 + 1\right)^2}{\left(1 - k^2\right)}$, or

$$b = \pm\sqrt{2}\frac{R^{-1}\left(k^2 + 1\right)}{\sqrt{1 - k^2}}. \qquad (3.7)$$

Substituting (3.7) into the third equation (3.5) we obtain

$$a = \pm 2R^{-1}\sqrt{b^{-1} - 1} \qquad (3.8)$$

Expressions (3.6)-(3.8) uniquely define two fixed stationary points of the system (3.4).

Numerical calculations show that the data fixed points are stable. Regardless of the choice of initial conditions, all solutions tend to one of the two fixed points. An example of numerical solution of the nonlinear system (3.4) is presented in Fig. 2.

The coordinates of the fixed points agree well with expressions (3.6)-(3.8). In particular at $k = 0.5$, $R^{-1} = 0.1$ the theoretical limits of (3.6)-(3.8) are: $b = 0.2054$, $a = -0.39$, $c = 0.806$. These values practically coincide with the values in the numerical calculation.

The proof of stability of fixed points can be done using a system for linear approximation. Let us denote the coordinates of the fixed points (3.6)-(3.8) as $b = b_0$, $c = c_0$, $a = a_0$. Let us put $a(t) = a_0 + a$, $b(t) = b_0 + b$, $c(t) = c_0 + c$. Then for the perturbations from (3.4) we have a linearized system

$$a_t(t)\left(1+k^2\right) + \left(b_0 c(t) + c_0 b\right) k \left(1-k^2\right) = -R^{-1}\left(k^2+1\right)^2 a,$$

$$b_t - (1/2) k (a_0 c + c_0 a) = -R^{-1} b,$$

$$c_t + (1/2) k (a_0 b + b_0 a) = -R^{-1} k^2 c \qquad (3.9)$$

Various analytical criteria concerning the behavior of perturbations can be used to prove linear stability. However, it is easier, however, to perform a direct numerical calculation of solutions of the system (3.9). An example of such a calculation is presented in Fig. 3. As can be seen, all solutions of the linearized system tend to zero, which indicates linear stability.

Thus, the calculation results show that, in the presence of friction, the development of nonlinear instability leads to the formation of a system of stationary closed vortex cells in a periodic zonal flow. In the presence of zonal flow, the full function of the flow current is determined by the expression $\psi = a(x,t)\cos y + b(x,t)\sin y + c(x,t)$, or, in the limiting case, for the values of $k = 0.5$, $R^{-1} = 0.1$

$$\psi = -0.39 \cos kx \cos y + 0.204 \sin y + 0.806 \sin kx. \qquad (3.10)$$

Isolines of the current function (3.10) are shown in Fig. 4. As can be seen, all isolines of this function are sloped along the flow.

An important result is that the modulus of the horizontally averaged velocity is less than the modulus of the main flow velocity. If we denote the horizontally averaged velocity by angle brackets, then for the main flow $\langle U \rangle = -1$, and for the flow with cells, according to (18), $\langle U \rangle = -0.204$. The formation of vortex cells thus leads to a decrease in the modulus of the mean flow velocity. As shown below, this feature is also preserved for the stratified fluid model.

The behavior described above with the establishment of stationary cells is fundamentally different from the oscillatory behavior in the absence of friction.

4. STABILITY PROBLEM FORMULATION FOR A SEMI-CONFINED STRATIFIED ATMOSPHERE

To study the stability of periodic currents, we use the equations of the surface geostrophic model (SQG-model) describing the motions of a stratified rotating fluid with zero potential vorticity [21, 22]. The model considers a stratified, rotating and semi-infinite atmosphere ($z > 0$) with a constant buoyancy frequency N and an inertial frequency f . Atmospheric motions with characteristic velocity U_0 , horizontal scale D and Rossby number $\mathrm{Ro} = U_0/fD \ll 1$ are considered. In dimensionless variables, the equations of the SQG model include the Laplace equation for the current function ψ in the inner region

$$\psi_{xx} + \psi_{yy} + \psi_{zz} = 0. \quad (4.1)$$

Here, the horizontal and vertical scales are shafted as D and $H = Df/N$, the time scale and the current function as $T = D/U_0$ and $\psi_0 = U_0 D$, respectively. Dimensionless horizontal components of velocity u, v and buoyancy perturbation σ (potential temperature) are related to the current function by the relations $u = -\psi_y,\ \mathrm{v} = \psi_x, \sigma = \psi_z$.

The equation (4.1) is supplemented by an important boundary condition

$$z = 0: \qquad \psi_{zt} + \left[\psi, \psi_z\right] = -r\Delta_2\psi + F \quad . \qquad (4.2)$$

Here square brackets denote the two-dimensional Jacobian on the variables $,x, y[m,n] = m_x n_y - m_y n_x$. Also denoted $r = \mathrm{E}^{1/2}/2\mathrm{Ro}$ is the bottom friction coefficient, $\mathrm{E} = \left(h_E/H\right)^2$ is the Ekman number, $h_E = \left(2\nu/f\right)^{1/2}$ is the thickness of the Ekman boundary layer with the effective turbulent toughness coefficient ν . The detailed conclusions of condition (4.2) are presented in the monograph [23]. Note that the friction coefficient can be represented as $r = \left(\frac{2\nu f}{U_*^2}\right)^{1/2} \frac{D}{H}$. Also note that for the given values of the parameters and $h_E = 0.5\text{км}$, the friction coefficient $r = 4$.

In condition (4.2) there is an external periodic force acting on the boundary F. Further we consider this force to be stationary and spatially periodic $F = -r\cos y$. In the absence of friction and external force, condition (4.2) is the equations of buoyancy transfer at the horizontal boundary.

Directly from (4.1), (4.2) follows the equation of the total energy balance

$$E_t = -r < \psi_x^2 + \psi_y^2 >\Big|_{z=0}, \ E = \int_0^1 < \left(\psi_x^2 + \psi_y^2 + \psi_z^2\right) > dz, \qquad (4.3)$$

reflecting the dissipative character of bottom friction. Here angular brackets denote the operation of averaging over horizontal coordinates.

We emphasize that for solutions of the Laplace equation (4.1) (harmonic functions), the values of the current function ψ at the boundaries are expressed through the boundary value of the normal derivative $\sigma = \psi_z$ (by means of a nonlocal Hilbert-type operator). The description of three-dimensional dynamics of currents with zero potential vorticity is thus reduced to the solution of the two-dimensional equation (4.2) at the boundary. This explains the term surface geostrophic model (SQG model). The model is described in detail in [22, 24-26].

The system (4.1), (4.2) in the presence of an external force $F = -r\cos y$ on the boundary has an exact solution

$$\Psi = e^{-z}\cos y, \qquad (4.4)$$

describing a zonal spatially periodic flow with velocity $U = -\Psi_y = e^{-z}\sin y$, buoyancy $\sigma = \Psi_z = -e^{-z}\cos y$, localized near the underlying surface. The dimensional shape of the velocity profile $U = U_0 e^{-z/H}\sin(y/D)$, $H = Df/N$. Representing $\psi = \Psi + \psi'$ and omitting the dash, we obtain the Laplace equation (4.1) with boundary condition to describe the perturbations

$$z = 0: \qquad \psi_{zt} + \sin y\left(\psi_{xz} + \psi_x\right) + \left[\psi, \psi_z\right] = -r\Delta_2\psi \quad , (4.5)$$

and the attenuation condition at $z \to \infty$.

Within the framework of problem (4.1), (4.5), linear and nonlinear dynamics of perturbations of the periodic flow (4.4) will be studied further. We emphasize that the nonlinear term of the problem is contained only in the boundary conditions and describes the nonlinear advection of the surface buoyancy field.

5. LINEAR STABILITY THEORY. GALERKIN METHOD

As before, we use the Galerkin method with three basis functions $f_1 = \sin y, f_2 = \cos y$, $f_3 = 1$ along the transverse coordinate y to describe the linear dynamics of the perturbations. According to this method, the approximate solution for the perturbations is sought in the form of expansion by basis functions

$$\psi = A(x,z,t)\sin y + B(x,z,t)\cos y + C(x,z,t). \qquad (5.1)$$

By virtue of the Laplace equation (4.1), the expansion coefficients satisfy the equations

$$A_{xx} + A_{zz} - A = 0, \; ,B_{xx} + B_{zz} - B = 0 C_{xx} + C_{zz} = 0. \qquad (5.2)$$

From the linearized form of condition (4.5), the equations relating the distributions A, B, C at the boundary also follow. To obtain these equations, we write the condition (4.5) in the form $z = 0: \;\; L(\psi) = \psi_{zt} + \sin y\left(\psi_{xz} + \psi_x\right) + r\Delta_2\psi = 0$ and use the orthogonality conditions

$\langle f_i(x)L(\psi)\rangle = 0,\ i = 1,2,3.$ Hereinafter, angle brackets denote the period averaging operation 2π: $\langle\phi\rangle = (2\pi)^{-1}\int_0^{2\pi}\phi\,dy$ Considering (4.5), we obtain

$$L(\psi) = A_{zt}\sin y + B_{zt}\cos y + C_{zt} + \sin y(A_{xz} + A_x)\sin y + (B_{xz} + B_x)\cos y + C_{xz} + C_x) + \\ +r((A_{xx} - A)\sin y + (B_{xx} - B)\cos y + C_{xx}) \quad (5.3)$$

Hence from the orthogonality conditions follow the equations

$A_{zt} + (C_{xz} + C_x) + r(A_{xx} - A) = 0\ , C_{zt} + (1/2)(A_{xz} + A_x) + rC_{xx} = 0$. (5.4)

These equations are joined by an isolated equation

$B_{zt} + r(B_{xx} - B) = 0,$ (5.5)

having only solutions damped with time, in particular $B = 0$. The conditions (5.4) are considered at z = 0.

Harmonic on coordinate$x$ solutions of equations (25) can be written in the form

$$A = a(t)e^{-k_1 z}\sin(kx),\quad C = c(t)e^{-kz}\cos(kx),\ B = 0, \quad (5.6)$$

where $k_1 = \sqrt{k^2 + 1}$. Substituting (5.6) into the conditions (5.4) at z = 0 leads to a system of linear ordinary differential equations

$$a_t(t) + k_1^{-1}k(1 - k)c(t) + rk_1^{-1}(k^2 + 1)a = 0, \quad (5.7)$$
$$c_t(t) + (1/2)(k_1 - 1)a(t) + rkc = 0\ .$$

This system has exponentially increasing solutions with time. Assuming $a = Ae^{\lambda t},\ c = Ce^{\lambda t}$ from (5.7) we obtain from (5.7) a system of linear homogeneous equations

$$\left(\lambda + rk_1^{-1}\left(k^2 + 1\right)\right)A + k_1^{-1}k(1 - k)C = 0,$$
$$(1/2)\left(k_1 - 1\right)A + (\lambda + rk)C = 0.$$

Equating the determinant of this system to zero, we obtain a quadratic equation for the ramp-up increment

$$\lambda^2 + rk_1^{-1}\left(\left(k^2 + 1\right) + rk\right)\lambda + r^2k_1^{-1}\left(k^2 + 1\right)k - (1/2)k_1^{-1}k(1 - k)\left(k_1 - 1\right) = 0. \quad (5.8)$$

The condition of equality to zero of the free term of this equation gives the boundary value$r$, determining the occurrence of instability

$$r^{-2} = r_c^{-2}(k) = \frac{2(k^2 + 1)}{(1 - k)(k_1 - 1)}.$$

Instability exists if $r^{-2} > r_c^{-2}(k)$, or, equivalently,$(1 - k)\left(k_1 - 1\right)/2\left(k^2 + 1\right) < r_c$. The graph of the dependence $r^{-2} = r_c^{-2}(k)$, is shown in Fig. 5. The values of k lying above this curve, which has the form of a potential pit, correspond to instability.

6. NONLINEAR DYNAMICS OF PERTURBATIONS

Let us now describe the nonlinear dynamics of perturbations in the presence of friction. For the description we also use the Galerkin method with three basis functions. The advantage of this method is that it is relatively easy to construct an approximate solution to this nonlinear problem.

As before, we will search for an approximate solution in the form of expansion (5.1), where the expansion coefficients satisfy the linear equations (5.2) following from the Laplace equation (4.1). To obtain the nonlinear equations at the boundary of $z = 0$, we transform condition (4.5). The linear part $L(\psi)$ of this condition is given by expression (5.3). Direct calculation of the nonlinear term gives

$$\left[\psi, \psi_z\right] = (1/2)\left(BA_z - AB_z\right)_x + \left(BC_{xz} - C_x B_z\right)\sin y + \left(C_x A_z - AC_{xz}\right)\cos y + F\left(\sin 2y, \cos 2y\right),$$

where $F(\sin 2y, \cos 2y)$ denotes a linear combination of trigonometric functions of the dual argument. Substituting this expression and $L(\psi)$ (5.3) into (4.5), with the subsequent use of orthogonality conditions, leads to a nonlinear equation system at the boundary

$$z = 0: \quad A_{zt} + \left(B+1\right)C_{xz} + \left(1 - B_z\right)C_x - r\left(A_{xx} - A\right) = 0\,,$$

$$C_{zt} + (1/2)\left((B+1)A_z + (1-B_z)A\right)_x + rC_{xx} = 0,$$

$$B_{zt} + C_x A_z - AC_{xz} + r(B_{xx} - B) = 0\,. \qquad (6.1)$$

The linearized version of (6.1) obviously reduces to the boundary equations (5.4).

We will search for approximate solutions of equations (4.1), (6.1) of the form

$$A = a(t)e^{-k_1 z}\sin(kx),\; C = c(t)e^{-kz}\cos(kx), \quad B = b(t)e^{-z}, \qquad (6.2)$$

where $k_1 = \sqrt{k^2+1}$. For the chosen form of solutions, equation (4.1) is exactly satisfied and the boundary equations (6.1) reduce to nonlinear ordinary differential equations without any approximation. As before, we introduce the operation of averaging over the horizontal coordinate $\langle\phi\rangle = \lim\limits_{L\to\infty} \frac{1}{L}\int_0^L \phi dx$, $L = \frac{2\pi}{k}$ and for the chosen form of the solution we calculate the averaged nonlinear equation term of the last equation (6.1): $\left\langle\left(C_x A_z - AC_{xz}\right)\right\rangle = (1/2)k(k_1 - k)ac.$ It follows from the last expression that the approximate solution for the coefficient$B$ should depend only on time and vertical coordinate, which is taken into account in (6.2). Denoting for brevity $b = b+1$, thus, we obtain a system of ordinary differential equations to describe the nonlinear dynamics of perturbations

$$a_t + \alpha\, bc + rk_1^{-1}\left(k^2+1\right)a = 0,$$

$$c_t + \gamma\, ba + rkc = 0,$$

$$b_t - \beta ac + r(b-1) = 0. \qquad (6.3)$$

Here it is labeled

$$\alpha = k(1-k)/k_1, \quad \gamma = (k_1 - 1)/2, \quad \beta = k(k_1 - k)/2. \qquad (6.4)$$

Note that in the absence of friction the nonlinear system (6.3) is analogous to the system describing the motion of a symmetric shaft in mechanics (or the motion of a fluid in an ellipsoidal cavity). The laws of conservation derived from (6.3) are

$$\frac{d}{dt}\left(\gamma a^2 - \alpha c^2\right) = 0\,, \quad \frac{d}{dt}\left(\beta a^2 + \alpha b^2\right) = 0,$$

allow us to analytically integrate the system [27-29]. The corresponding solution describes nonlinear oscillations similar to those described earlier for the barotropic model.

An important feature of the nonlinear system (6.3) in the model with friction is that the stationary version of the system

$$\alpha\, bc + rk_1^{-1}\left(k^2 + 1\right)a = 0, \qquad \gamma\, ba + rkc = 0, \qquad -\beta ac + r(b-1) = 0\ . \quad (6.5)$$

has an exact stationary solution. Secondly, from the second equation (6.3) follows $c = -\gamma\, ba\, /\, rk$. Substituting this expression into the first equation, after reduction by a, shaft $b^2 = r^2 k_1^{-1}\, k\left(k^2 + 1\right)/\alpha\gamma$, or $b = \pm r\sqrt{k_1^{-1} k\left(k^2 + 1\right)/\alpha\gamma}$. Now let us substitute the value of c into the last equation. We obtain $a^2 = r^2 k\left(b^{-1} - 1\right)/\beta\gamma$, or $a = \pm r\sqrt{k\left(b^{-1} - 1\right)/\beta\gamma}$. These expressions uniquely determine the coordinates of the two stationary points of the system (6.3). For the values $r = 0.115,\ \ k = 0.6$, the calculations give $b = \pm 0.736, \quad a = \pm 0.45, \quad c = \pm 0.40$.

As calculations show, at small but finite values of r instead of oscillations, the regime of establishment of stationary periodic solutions with amplitudes (fixed points) following from the system (6.4) is realized. An example of the numerical solution of the nonlinear system (6.3) for the values $r = 0.115,\ \ k = 0.6$ and initial conditions $a(0) = 0.5,\ c(0) = 0.5,\ b(0) = 1$ is shown in Fig. 6.

The results of calculations show that the specified fixed points are stable. Regardless of the choice of initial conditions, all solutions of the system tend to one of the two fixed points. This indicates the stability of the fixed points without analytical stability criteria.

Thus, the development of nonlinear instability leads to the formation of a system of closed stationary vortex cells in a periodic zonal flow. In the presence of cells, the function of the flow current is determined by the expression $\psi = a(t)e^{-\kappa z}\sin(kx)\sin y + b(t)e^{-z}\cos y + c(t)e^{-kz}\cos(kx)$ or, in the limiting case, at the lower boundary z = 0 for the values of the parameters $r = 0.115,\ \ k = 0.6$

$$\psi = -0.45\sin(kx)\sin y + 0.736\cos y + 0.4\cos(kx). \qquad (6.6)$$

Isolines of the current function (6.6) are shown in Fig. 7. As in the model with no stratification, the isolines of this function are sloped along the flow. Stratification leads to a decrease in the length of the limiting S-waves in the transverse direction.

As before, the modulus of the horizontally averaged velocity of the flow in the presence of cells is less than the modulus of the main flow velocity. If we denote by angle brackets the horizontally averaged velocity, then for the main flow $\langle U \rangle = 1$, and for the flow with cells, according to (6.6), $\langle U \rangle = 0.736$. The formation of vortex cells, again, leads to a decrease in the modulus of the mean flow velocity.

We emphasize that we used the Galerkin model with three modes to describe the nonlinear dynamics of perturbations. As shown in [27], an increase in the number of modes does not lead to qualitatively new results. The model with three modes also provides a good approximation for the numerical values of the critical parameters.

It should also be noted that the currents of the stratified medium of periodic or quasiperiodic structure are often enough observed in the atmospheres of the planets. Thus, according to observation data, the distribution of the zonal current velocity by latitude in the atmosphere of Jupiter is practically periodic.

CONCLUSION

The stability of spatially periodic flows of homogeneous and stratified fluid with consideration of bottom friction is investigated. The Galerkin method with three basis Fourier-harmonics is used for the solution of the stability problem. A system of ordinary differential equations for the amplitudes of the Fourier-harmonics is formulated. On the basis of numerical solution of this system, it was shown in previous works of the authors that in the frictionless model the development of instability of flows leads to the regime of establishing oscillations or shafts. In this work it is shown that the situation changes fundamentally when friction is taken into account. Exponential growth of linear perturbations at the nonlinear stage of development is replaced by the regime of establishment of stationary periodic perturbations. These perturbations decrease the averaged horizontal flow velocity. Analytical expressions for the spatial period and amplitude of perturbations are obtained.

The obtained theoretical results for the Kolmogorov flow agree with the description of the experimental results presented in [16]; after passing the critical value of the Reynolds number R, the unidirectional flow becomes unstable, and a secondary flow in the form of a regular system of stationary shafts appears. As R increases, the stationary flow becomes unstable and periodic oscillations appear. Thus, the results of a rather simple theoretical model with three modes agree well with experiment.

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FIGURE CAPTIONS

Fig. 1. Nonlinear perturbation shafts (functions $a(t), b(t), c(t)$) in the frictionless model.

Fig. 2. Example of numerical solution of the system (12) at $k = 0.5$, $R^{-1} = 0.1$ with initial conditions $a(0) = 0.5$, $b(0) = 1$, $c(0) = 0.9$.

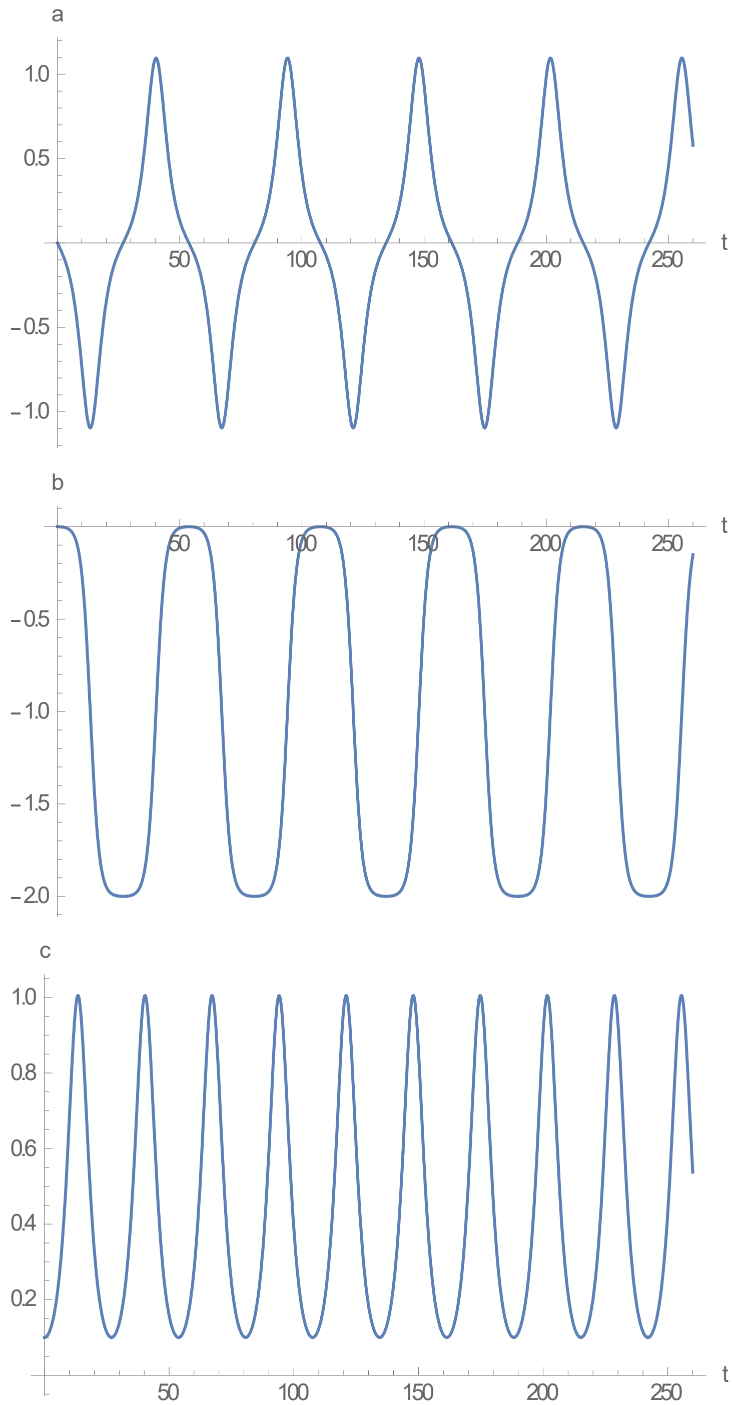
Fig. 3. Example of numerical solution of the linearized system.

Fig. 4. Isolines of the current function (18) established as a result of instability. The *x* and *y* coordinates are plotted along the horizontal and vertical axes.

Fig. 5. Neutral stability curve.

Fig. 6. Example of numerical solution of the system (34) for the values $r = 0.115$, $k = 0.6$ and initial conditions $a(0) = 0.5$, $c(0) = 0.5$, $b(0) = 1$.

Fig. 7. Isolines of the current function (37) establishing as a result of instability.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t
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100
150
200
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b
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-1.5
-2.0
t
50
100
150
200
250
c
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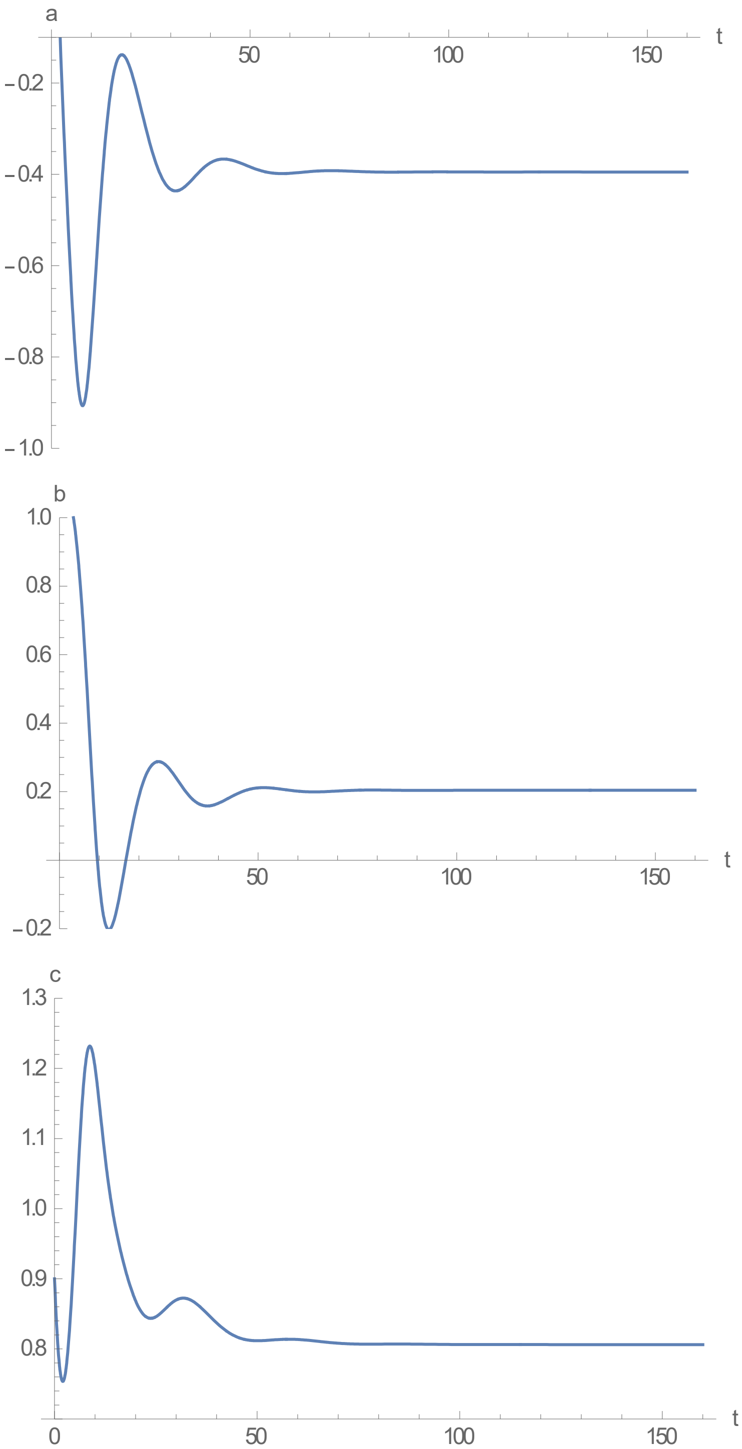
0.6
0.4
0.2
t
50
100
150
200
250

Fig. 1.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0.4
0.2
t
50
100
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t
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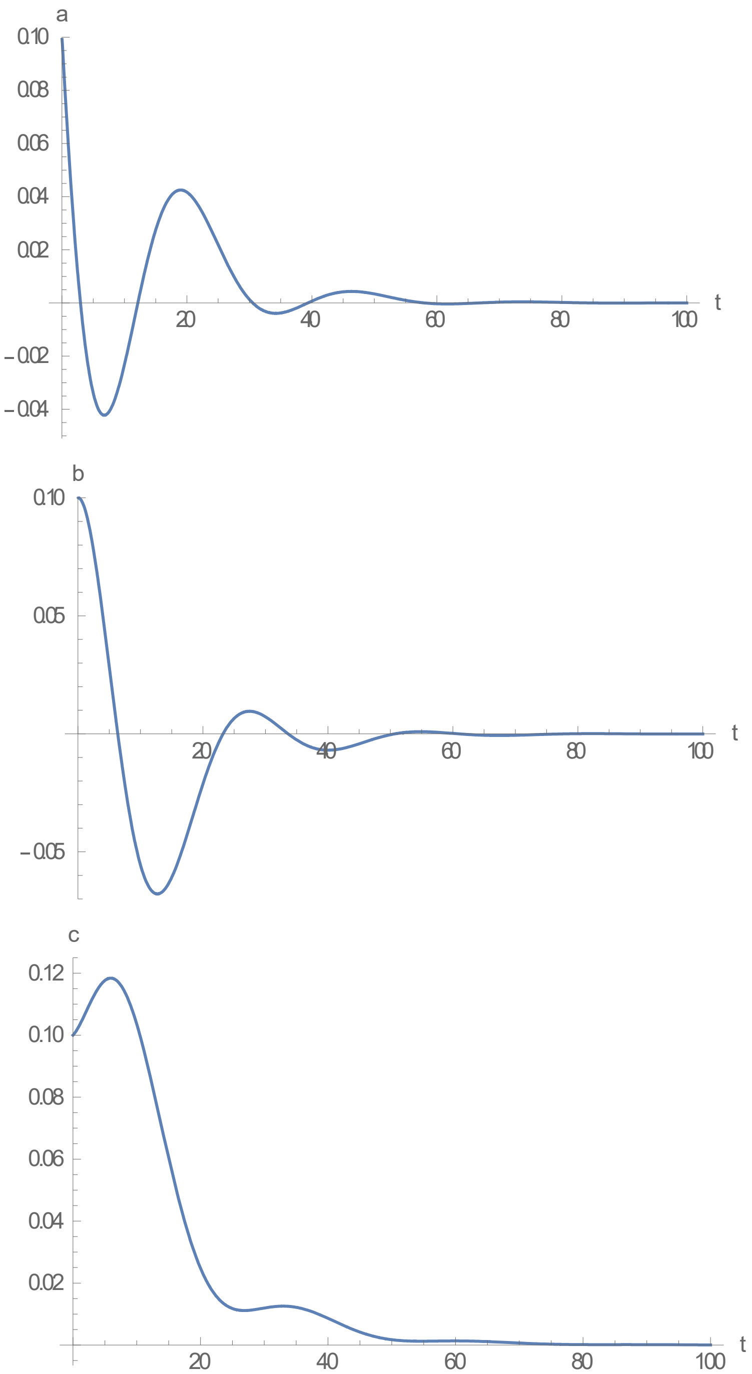
50
100
150

Fig. 2.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20
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60
80
100
t
b
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0.05
−0.05
20
40
60
80
100
t
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0.02
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40
60
80
100
t

Fig. 3.

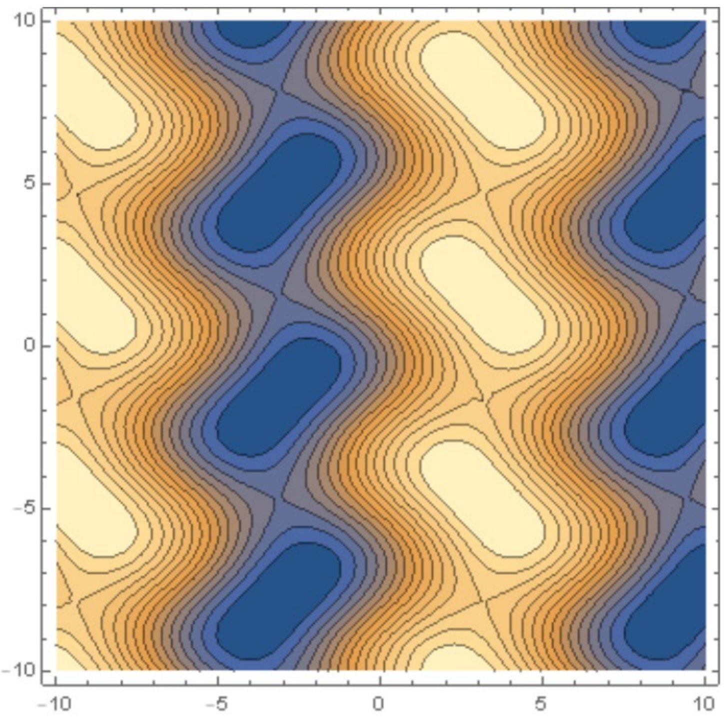
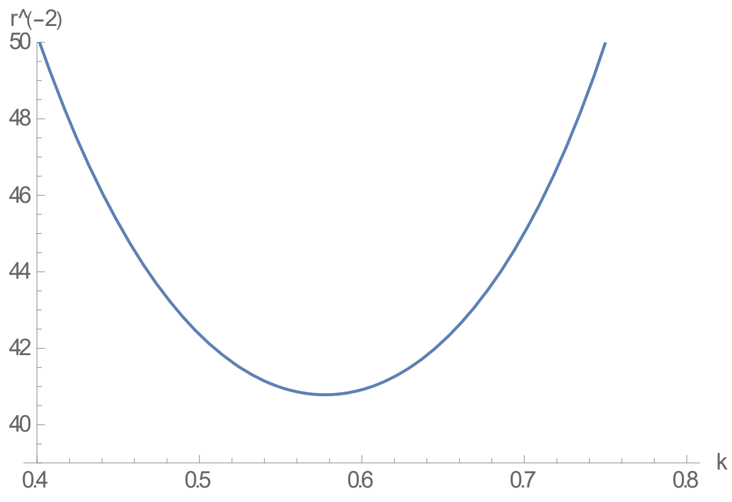


Fig. 4.



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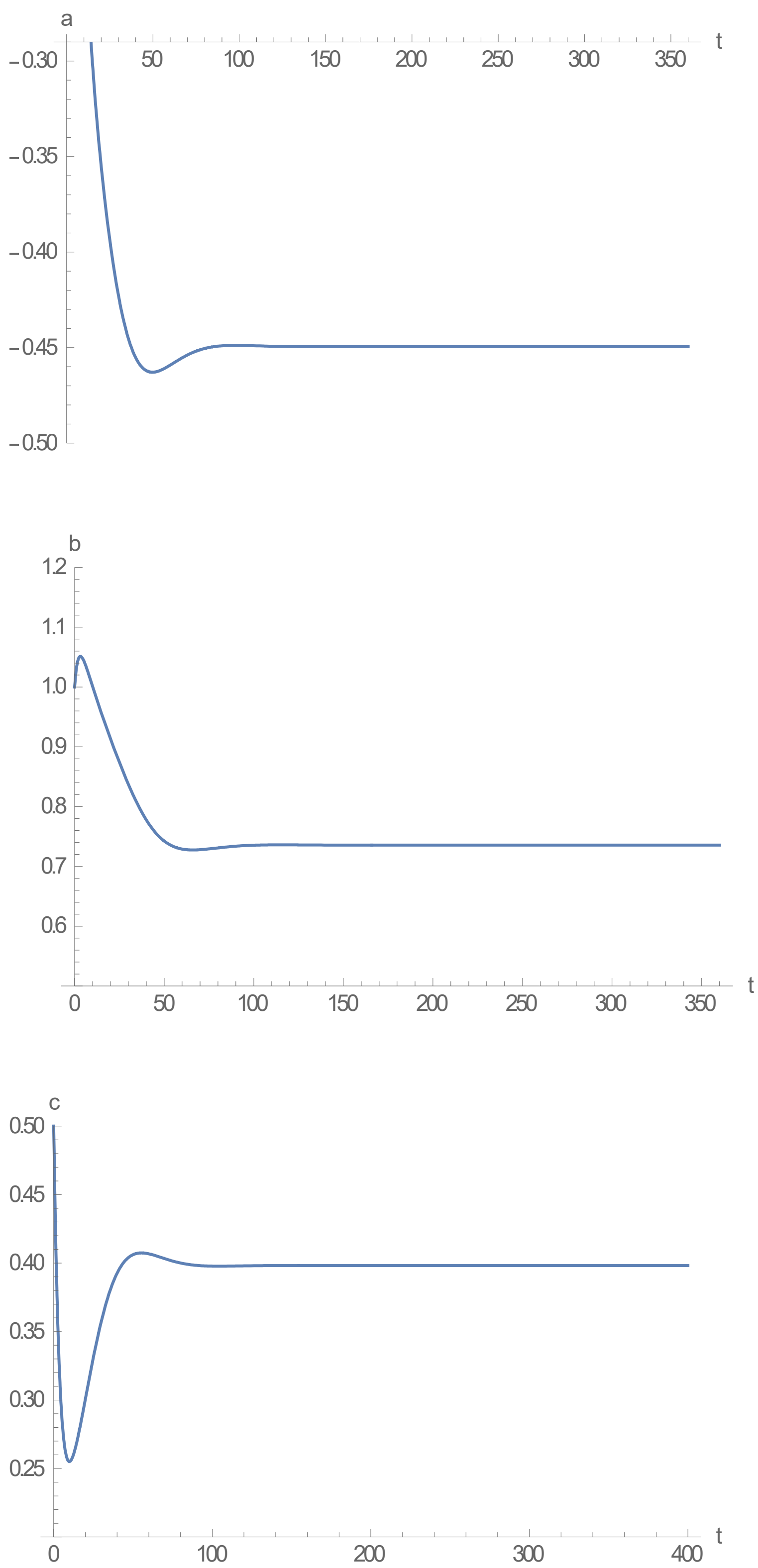
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50
100
150
200
250
300
350
b
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1.1
1.0
0.9
0.8
0.7
0.6
t
0
50
100
150
200
250
300
350
c
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0.40
0.35
0.30
0.25
t
0
100
200
300
400

Fig. 6.

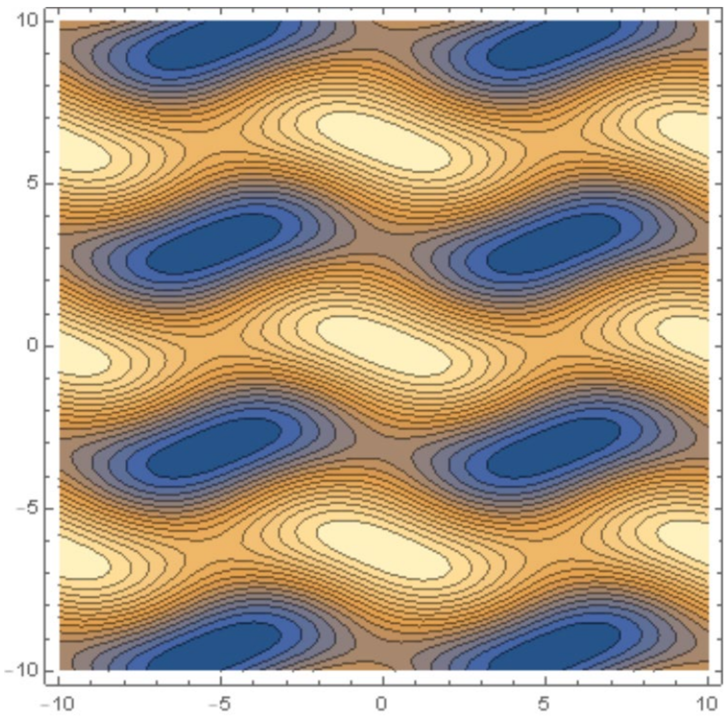

Fig. 7.