Supplementary Material to the article "Valley Hall effect in two-dimensional electron-exciton system"

1. Full scattering integral for the skew collisions between electrons and bogolons $Q^{as}=Q^{as}_{(a)}+Q^{as}_{(b,c)}$ reads

$$\begin{gathered}
Q^{as}_{(a)}\{n_{\mathbf{p}}\}=\frac{\gamma^2 n_c}{E_g^2(2\pi)^2}\int d\mathbf{p}_1 d\mathbf{p}_2 g_{\mathbf{p}-\mathbf{p}_1}g_{\mathbf{p}_1-\mathbf{p}_2}g_{\mathbf{p}_2-\mathbf{p}}\\
(\mathbf{p}\times\mathbf{p}_1+\mathbf{p}_1\times\mathbf{p}_2+\mathbf{p}_2\times\mathbf{p})_z(\tilde{u}_{\mathbf{p}-\mathbf{p}_1}+\tilde{v}_{\mathbf{p}-\mathbf{p}_1})(\tilde{u}_{\mathbf{p}-\mathbf{p}_2}+\tilde{v}_{\mathbf{p}-\mathbf{p}_2})\\
\Big\{\Big[(1-n_{\mathbf{p}})n_{\mathbf{p}_1}N_{\mathbf{p}-\mathbf{p}_1}-n_{\mathbf{p}}(1-n_{\mathbf{p}_1})(1+N_{\mathbf{p}-\mathbf{p}_1})\Big]\chi^T_{\mathbf{p}-\mathbf{p}_1}\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}_1}-\omega_{\mathbf{p}-\mathbf{p}_1})+\\
+\Big[(1-n_{\mathbf{p}})n_{\mathbf{p}_1}(1+N_{\mathbf{p}-\mathbf{p}_1})-n_{\mathbf{p}}(1-n_{\mathbf{p}_1})N_{\mathbf{p}-\mathbf{p}_1}\Big]\tilde{\chi}^T_{\mathbf{p}-\mathbf{p}_1}\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}_1}+\omega_{\mathbf{p}-\mathbf{p}_1})\Big\}\\
\Big\{(1+N_{\mathbf{p}-\mathbf{p}_2}-n_{\mathbf{p}_2})\chi_{\mathbf{p}-\mathbf{p}_2}\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}_2}-\omega_{\mathbf{p}-\mathbf{p}_2})+(N_{\mathbf{p}-\mathbf{p}_2}+n_{\mathbf{p}_2})\tilde{\chi}_{\mathbf{p}-\mathbf{p}_2}\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}_2}+\omega_{\mathbf{p}-\mathbf{p}_2})\Big\},
\end{gathered}\tag{S1}$$

$$\begin{gathered}
Q^{as}_{(b,c)}\{n_{\mathbf{p}}\}=\frac{\gamma^2 n_c}{E_g^2(2\pi)^2}\int d\mathbf{p}_1 d\mathbf{p}_2 g_{\mathbf{p}-\mathbf{p}_1}g_{\mathbf{p}_1-\mathbf{p}_2}g_{\mathbf{p}_2-\mathbf{p}}\\
(\mathbf{p}\times\mathbf{p}_1+\mathbf{p}_1\times\mathbf{p}_2+\mathbf{p}_2\times\mathbf{p})_z(\tilde{u}_{\mathbf{p}-\mathbf{p}_1}+\tilde{v}_{\mathbf{p}-\mathbf{p}_1})(\tilde{u}_{\mathbf{p}_1-\mathbf{p}_2}+\tilde{v}_{\mathbf{p}_1-\mathbf{p}_2})\\
\Big\{(1-n_{\mathbf{p}})\Big[\chi^T_{\mathbf{p}-\mathbf{p}_1}N_{\mathbf{p}-\mathbf{p}_1}\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}_1}-\omega_{\mathbf{p}-\mathbf{p}_1})+\tilde{\chi}^T_{\mathbf{p}-\mathbf{p}_1}(1+N_{\mathbf{p}-\mathbf{p}_1})\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}_1}+\omega_{\mathbf{p}-\mathbf{p}_1})\Big]\\
\Big[\big(n_{\mathbf{p}_1}(1+N_{\mathbf{p}_1-\mathbf{p}_2}-n_{\mathbf{p}_2})-n_{\mathbf{p}_2}N_{\mathbf{p}_1-\mathbf{p}_2}\big)\chi_{\mathbf{p}_1-\mathbf{p}_2}\delta(\varepsilon_{\mathbf{p}_1}-\varepsilon_{\mathbf{p}_2}-\omega_{\mathbf{p}_1-\mathbf{p}_2})+\\
+\big(n_{\mathbf{p}_1}(N_{\mathbf{p}_1-\mathbf{p}_2}+n_{\mathbf{p}_2})-n_{\mathbf{p}_2}(1+N_{\mathbf{p}_1-\mathbf{p}_2})\big)\chi_{\mathbf{p}_1-\mathbf{p}_2}\delta(\varepsilon_{\mathbf{p}_1}-\varepsilon_{\mathbf{p}_2}+\omega_{\mathbf{p}_1-\mathbf{p}_2})\Big]-\\
-n_{\mathbf{p}}\Big[\chi^T_{\mathbf{p}-\mathbf{p}_1}(1+N_{\mathbf{p}-\mathbf{p}_1})\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}_1}-\omega_{\mathbf{p}-\mathbf{p}p_1})+\tilde{\chi}^T_{\mathbf{p}-\mathbf{p}_1}N_{\mathbf{p}-\mathbf{p}_1}\delta(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}_1}+\omega_{\mathbf{p}-\mathbf{p}_1})\Big]\\
\Big[\big((1-n_{\mathbf{p}_1})(1+N_{\mathbf{p}_1-\mathbf{p}_2}-n_{\mathbf{p}_2})-(1-n_{\mathbf{p}_2})N_{\mathbf{p}_1-\mathbf{p}_2}\big)\chi_{\mathbf{p}_1-\mathbf{p}_2}\delta(\varepsilon_{\mathbf{p}_1}-\varepsilon_{\mathbf{p}_2}-\omega_{\mathbf{p}_1-\mathbf{p}_2})+\\
+\big((1-n_{\mathbf{p}_1})(N_{\mathbf{p}_1-\mathbf{p}_2}+n_{\mathbf{p}_2})-(1-n_{\mathbf{p}_2})(1+N_{\mathbf{p}_1-\mathbf{p}_2})\big)\chi_{\mathbf{p}_1-\mathbf{p}_2}\delta(\varepsilon_{\mathbf{p}_1}-\varepsilon_{\mathbf{p}_2}+\omega_{\mathbf{p}_1-\mathbf{p}_2})\Big]\Big\},
\end{gathered}\tag{S2}$$

where

$$\begin{aligned}
\chi^T_{\mathbf{p}}&=(\tilde{u}_{\mathbf{p}},\tilde{v}_{\mathbf{p}}),\\
\tilde{\chi}^T_{\mathbf{p}}&=(\tilde{v}_{\mathbf{p}},\tilde{u}_{\mathbf{p}}),
\end{aligned}\tag{S3}$$

and the evenness of amplitudes $\tilde{u}$, $\tilde{v}$ and bogolon dispersion is taken into account:

$$\begin{gathered}
\tilde{u}_{-\mathbf{p}}=\tilde{u}_{\mathbf{p}},\quad \tilde{v}_{-\mathbf{p}}=\tilde{v}_{\mathbf{p}},\\
\omega_{-\mathbf{p}}=\omega_{\mathbf{p}},\quad N_{-\mathbf{p}}=N_{\mathbf{p}}.
\end{gathered}\tag{S4}$$

2. Let us clarify the derivation of the **self-energies** Σ (Fig.2 of the main text). For instance, the symmetrical diagram $\Sigma^{(a)}$ is constructed by vertexes $\hat{V}_1(x_1,y_1)\hat{V}_2(x_2,y_2)\hat{V}_1(x_3,y_3)$, which give the exciton part of $\Sigma^{(a)}$:

$$\langle T_C(\hat{\phi}^\dagger(y_1)+\hat{\phi}(y_1))\hat{\phi}^\dagger(y_2)\hat{\phi}(y_2)(\hat{\phi}^\dagger(y_3)+\hat{\phi}(y_3))\rangle,\tag{S5}$$

where $\langle\cdots\rangle$ denotes the equilibrium average of bose-condensed subsystem and T_C is the operator of time-ordering at Keldysh contour. Exp. (S5) is evaluated with Wick's theorem, which results in the set of products of different Green's functions. It is convenient to write these functions in matrix form:

$$\hat{P}(y_1,y_2)=\begin{pmatrix}\mathcal{G}(y_1,y_2) & \mathcal{F}(y_1,y_2)\\ \tilde{\mathcal{F}}(y_1,y_2) & \tilde{\mathcal{G}}(y_1,y_2)\end{pmatrix}=-i\begin{pmatrix}\langle T_C\hat{\phi}(y_1)\hat{\phi}^\dagger(y_2)\rangle & \langle T_C\hat{\phi}(y_1)\hat{\phi}(y_2)\rangle\\ \langle T_C\hat{\phi}^\dagger(y_1)\hat{\phi}^\dagger(y_2)\rangle & \langle T_C\hat{\phi}^\dagger(y_1)\hat{\phi}(y_2)\rangle\end{pmatrix}. \tag{S6}$$

Thus Exp.(S5) becomes

$$(\tilde{\mathcal{F}}_{12}+\mathcal{G}_{12})(\mathcal{G}_{23}+\mathcal{F}_{23})+(\tilde{\mathcal{G}}_{12}+\mathcal{F}_{12})(\tilde{\mathcal{F}}_{23}+\tilde{\mathcal{G}}_{23}). \tag{S7}$$

For further computation we used model of weakly interacting Bose gas:

$$\hat{P}_{\mathbf{p},\omega}=\begin{pmatrix}\tilde{u}_{\mathbf{p}}^2 & \tilde{u}_{\mathbf{p}}\tilde{v}_{\mathbf{p}}\\ \tilde{u}_{\mathbf{p}}\tilde{v}_{\mathbf{p}} & \tilde{v}_{\mathbf{p}}^2\end{pmatrix}b_{\mathbf{p},\omega}+\begin{pmatrix}\tilde{v}_{\mathbf{p}}^2 & \tilde{u}_{\mathbf{p}}\tilde{v}_{\mathbf{p}}\\ \tilde{u}_{\mathbf{p}}\tilde{v}_{\mathbf{p}} & \tilde{u}_{\mathbf{p}}^2\end{pmatrix}c_{\mathbf{p},\omega}, \tag{S8}$$

where

$$\begin{aligned}
b^R_{\mathbf{p},\omega}&=\frac{1}{\omega-\omega_{\mathbf{p}}+i\delta},\quad b^<_{\mathbf{p},\omega}=-2\pi iN_{\mathbf{p}}\delta(\omega-\omega_{\mathbf{p}}),\quad b^>_{\mathbf{p},\omega}=-2\pi i(1+N_{\mathbf{p}})\delta(\omega-\omega_{\mathbf{p}}),\\
c^R_{\mathbf{p},\omega}&=\frac{1}{\omega+\omega_{\mathbf{p}}+i\delta},\quad c^<_{\mathbf{p},\omega}=-2\pi i(1+N_{\mathbf{p}})\delta(\omega+\omega_{\mathbf{p}}),\quad c^>_{\mathbf{p},\omega}=-2\pi iN_{\mathbf{p}}\delta(\omega+\omega_{\mathbf{p}}).
\end{aligned}$$